SHEAFIFICATION OF LINEAR FUNCTORS

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Abstract. For a stack $\mathcal{X}$ over a ring R and a fixed collection of quasi-coherent sheaves $\mathcal{C}$ on $\mathcal{X}$ we define a "sheafification" functor from the category of contravariant R-linear functors $\mathcal{C} \to \mathrm{Mod}(R)$ to the category of quasi-coherent sheaves on $\mathcal{X}$. When $\mathcal{X}$ is a projective scheme and $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}(n)\}_{n\in\mathbb{Z}}$ this sheafification operation coincides with the classical sheafification of graded modules over the homogeneous coordinate ring of $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1))$.

We consider many properties of the classical sheafification and extend them to our general setting: this is always possible when $\mathcal{C}$ generates $\mathrm{QCoh}(\mathcal{X})$. This theory finds applications in the area of Galois covers, in the study of certain stacks of fiber functors, in the construction of colimits of stacks and allows a generalization of Drinfeld's lemma from projective to proper schemes.

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Introduction

If $\mathcal{X}$ is a projective scheme over a ring R with very ample invertible sheaf $\mathcal{O}_{\mathcal{X}}(1)$ then quasi-coherent sheaves on $\mathcal{X}$ can be interpreted as graded modules over the homogeneous coordinate ring $S_{\mathcal{X}}$ of $\mathcal{X}$. If $\mathrm{GMod}(-)$ denote the category of graded modules then the Serre functor

$$\Gamma_* \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{GMod}(S_{\mathcal{X}}),\ \Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^0(\mathcal{X}, \mathcal{F}(n))$$

is fully faithful. Moreover there is an inverse operation, called homogeneous sheafification, which defines an exact functor $\widetilde{-} \colon \mathrm{GMod}(S_{\mathcal{X}}) \to \mathrm{QCoh}(\mathcal{X})$, left adjoint to Γ_*, and such that $\widetilde{\Gamma_*(\mathcal{G})} \to \mathcal{G}$ is an isomorphism for all $\mathcal{G} \in \mathrm{QCoh}(\mathcal{X})$.

Let us consider now a very different situation. Let G be an affine group scheme over a field k. Classical Tannaka's reconstruction problem consists in reconstructing the group G and more generally G-torsors from the category of representations $\mathrm{Rep}^G k$: a G-torsor $\pi \colon P \to S$ is completely determined by the associated exact *strong* monoidal functor

$$(\pi_*\mathcal{O}_P \otimes -)^G \colon \mathrm{Rep}^G k \to \mathrm{Vect}(S)$$

where $\mathrm{Vect}(S)$ denotes the category of locally free sheaves on S.

When G is finite, in [Ton14] I have introduced the notion of G-cover $f : X \to S$ extending the notion of G-torsor and, in order to handle the non abelian case, I showed in my Ph.D. thesis [Ton13] that those objects can also be reconstructed from the exact and *lax* monoidal functors $(f_*\mathcal{O}_X \otimes -)^G \colon \mathrm{Rep}^G k \to \mathrm{Vect}(S)$. The idea is that a G-cover is a weak version of a G-torsor, therefore we have to look for a weak version of a strong monoidal functor, that is, as the words suggest, a lax monoidal functor.

In this paper we develop a theory of sheafification functors which generalizes the two above situations. Let us introduce some notations and definitions. We fix a base commutative ring R and a category fibered in groupoids $\mathcal{X}$ over R. We say that $\mathcal{X}$ is *pseudo-algebraic* (resp. *quasi-compact*) if there exist a scheme (resp. an affine scheme) X and a map $X \to \mathcal{X}$ representable by fpqc covering of algebraic spaces. We denote by $\mathrm{QCoh}\,\mathcal{X}$ the category of quasi-coherent sheaves on $\mathcal{X}$ (see Section 1). Given a full subcategory $\mathcal{C}$ of $\operatorname{QCoh}\mathcal{X}$ we say that $\mathcal{C}$ generates $\operatorname{QCoh}\mathcal{X}$ if all quasi-coherent sheaves on $\mathcal{X}$ are quotient of a (possibly infinite) direct sum of sheaves in $\mathcal{C}$.

In what follows A will denote an R-algebra and $\mathcal{C}$ a full subcategory of $\operatorname{QCoh}\mathcal{X}$. We define the *Yoneda functors* associated with $\mathcal{C}$ as the R-linear contravariant functors

$$\Omega^{\mathcal{F}}\colon \mathcal{C} \to \operatorname{Mod} A,\ \Omega^{\mathcal{F}}_{\mathcal{E}} = \operatorname{Hom}_{\mathcal{X}}(\pi^*\mathcal{E},\mathcal{F}) \text{ for } \mathcal{F} \in \operatorname{QCoh}(\mathcal{X}_A)$$

where $\pi\colon \mathcal{X}_A = \mathcal{X} \times_R A \to \mathcal{X}$ is the projection. We also denote by $\mathrm{L}_R(\mathcal{C}, A)$ the category of contravariant R-linear functors $\mathcal{C} \to \operatorname{Mod} A$, so that we obtain a functor

$$\Omega^*\colon\ \operatorname{QCoh}(\mathcal{X}_A) \to \mathrm{L}_R(\mathcal{C}, A)$$

When $R = k$, $\mathcal{X} = \mathrm{B}\,G$, $\mathcal{C} = \operatorname{Vect}\mathcal{X} = \operatorname{Rep}^G k$ and $f : X \to \operatorname{Spec} A$ is a G-torsor or a G-cover one has that $f_*\mathcal{O}_{\mathcal{X}}$ can be thought of as a quasi-coherent sheaf of algebras over $\mathrm{B}\,G \times A$ and $\Omega^{f_*\mathcal{O}_X} = \operatorname{Hom}_{\mathrm{B}\,G}(-, f_*\mathcal{O}_X) = (f_*\mathcal{O}_X \otimes (-)^\vee)^G$. Up to a dual, this is the functor we associated with a G-cover in the above discussion.

Now consider the case when $\mathcal{X}$ is a projective scheme over a ring R with ample invertible sheaf $\mathcal{O}_{\mathcal{X}}(1)$ and coordinate ring $S_{\mathcal{X}}$. Set $\mathcal{C}_{\mathcal{X}} = \{\mathcal{O}_{\mathcal{X}}(n)\}_{n\in\mathbb{Z}}$. It is easy to realize that $\mathrm{L}_R(\mathcal{C}_{\mathcal{X}}, R)$ is equivalent to $\operatorname{GMod}(S_{\mathcal{X}})$ and that $\Omega^*\colon\ \operatorname{QCoh}(\mathcal{X}) \to \mathrm{L}_R(\mathcal{C}, R)$ is just the Serre functor $\Gamma_*\colon\ \operatorname{QCoh}(\mathcal{X}) \to \operatorname{GMod}(\mathcal{X})$ we started with (see 7 for details).

Once we have found this common language some questions arise naturally:

- Is the functor $\Omega^*\colon\ \operatorname{QCoh}(\mathcal{X}_A) \to \mathrm{L}_R(\mathcal{C}, A)$ fully faithful like the Serre functor?
- Does it also admits a left adjoint with the nice property of the sheafification functor?
- In the context of Tannaka duality sheaves of algebras yield (lax) monoidal functors. Is it true in general?
- Which functors are in the essential image of Ω^*?

For the second problem, it turns out quite easily that also $\Omega^*\colon\ \operatorname{QCoh}(\mathcal{X}_A) \to \mathrm{L}_R(\mathcal{C}, A)$ has a left adjoint

$$\mathcal{F}_{*,\mathcal{C}}\colon \mathrm{L}_R(\mathcal{C}, A) \to \operatorname{QCoh}(\mathcal{X}_A)$$

when $\mathcal{C}$ is essentially small: by analogy we call this functor a *sheafification* functor, which also explains the name of the paper.

The third problem is also easy. If $\mathcal{C}$ is a monoidal subcategory of $\operatorname{QCoh}\mathcal{X}$, $\mathrm{ML}_R(\mathcal{C}, A)$ denotes the category of R-linear, contravariant and (lax) monoidal functors $\mathcal{C} \to \operatorname{Mod} A$ and $\operatorname{QAlg}(\mathcal{X})$ the category of quasi-coherent sheaves of algebras on $\mathcal{X}$ then Ω^* and $\mathcal{F}_{*,\mathcal{C}}$ extend to a pair of left adjoint functors

$$\Omega^*\colon\ \operatorname{QAlg}(\mathcal{X}_A) \to \mathrm{ML}_R(\mathcal{C}, A) \text{ and } \mathscr{A}_{*,\mathcal{C}}\colon\ \mathrm{ML}_R(\mathcal{C}, A) \to \operatorname{QAlg}(\mathcal{X}_A)$$

Coming back to the fully faithfulness of Ω^* we prove the following.

Theorem A (4.6,4.12)**.** *Let $\mathcal{X}$ be a pseudo-algebraic category fibered in groupoids over R and $\mathcal{C} \subseteq \operatorname{QCoh}\mathcal{X}$ be a full subcategory generating $\operatorname{QCoh}\mathcal{X}$. Then the functor $\Omega^*\colon\ \operatorname{QCoh}(\mathcal{X}_A) \to \mathrm{L}_R(\mathcal{C}, A)$ is fully faithful and, if $\mathcal{C}$ is essentially small, the functor $\mathcal{F}_{*,\mathcal{C}}\colon \mathrm{L}_R(\mathcal{C}, A) \to \operatorname{QCoh}(\mathcal{X}_A)$ is exact and the natural map $\mathcal{G} \to \mathcal{F}_{\Omega^{\mathcal{G}},\mathcal{C}}$ is an isomorphism.*

If $\mathcal{X}$ is a projective scheme over R then it is a classical fact that $\mathcal{C}_{\mathcal{X}} = \{\mathcal{O}_{\mathcal{X}}(n)\}_{n\in\mathbb{Z}}$ generates $\operatorname{QCoh} X$ and thus we recover the classical properties of $\Gamma_*\colon\ \operatorname{QCoh}(\mathcal{X}_A) \to \operatorname{GMod}(S_{\mathcal{X}} \otimes_R A)$. We moreover extend this to quasi-compact and quasi-projective schemes over R and to certain quotient stacks (see 7.2).

Theorem above when $\mathcal{C}$ consists of a single object is a rephrasing of classical Gabriel-Popescu's theorem for the category $\operatorname{QCoh}\mathcal{X}$ (see 6.1). When $\mathcal{C}$ is monoidal and generates $\operatorname{QCoh}\mathcal{X}$ we also have that $\Omega^*\colon\ \operatorname{QAlg}(\mathcal{X}_A) \to \mathrm{ML}_R(\mathcal{C}, A)$ is fully faithful (see 6.8).

The last problem we address is to describe the essential image of Ω^*. The main idea is just that $\mathrm{Hom}_{\mathcal{X}}(-,\mathcal{F})$ for $\mathcal{F}\in\mathrm{QCoh}\,\mathcal{X}$ is a left exact functor. Since the domain $\mathcal{C}$ of the functors $\Omega^{\mathcal{F}}$ is not abelian we need an ad hoc definition of exactness. A *test sequence* in $\mathcal{C}$ is an exact sequence (in the ambient category $\mathrm{QCoh}\,\mathcal{X}$)

$$\mathcal{T}_*\colon \bigoplus_{k\in K}\mathcal{E}_k\to\bigoplus_{i\in I}\mathcal{E}_i\to\mathcal{E}\to 0 \text{ where } \mathcal{E},\mathcal{E}_i,\mathcal{E}_k\in\mathcal{C} \text{ and } I,J \text{ are sets}$$

such that $\alpha(\mathcal{E}_k)$ is contained in a finite sum for all $k\in K$. A test sequence is called finite if K and I are finite sets. Given $\Gamma\in\mathrm{L}_R(\mathcal{C},A)$ we will say that Γ is exact on a test sequence $\mathcal{T}_*$ in $\mathcal{C}$ if the complex of A-modules (see 4.3)

$$0\to\Gamma_{\mathcal{E}}\to\prod_{i\in I}\Gamma_{\mathcal{E}_i}\to\prod_{k\in K}\Gamma_{\mathcal{E}_k}$$

is exact. We denote by $\mathrm{Lex}_R(\mathcal{C},A)$ (resp. $\mathrm{MLex}_R(\mathcal{C},A)$ if $\mathcal{C}$ is monoidal) the full subcategory of $\mathrm{L}_R(\mathcal{C},A)$ (resp. $\mathrm{ML}_R(\mathcal{C},A)$) of functors which are exact an all test sequences. We have the following:

Theorem B (4.12, 5.7)**.** *Let $\mathcal{X}$ be a pseudo-algebraic category fibered in groupoids over R and $\mathcal{C}\subseteq\mathrm{QCoh}\,\mathcal{X}$ be a full subcategory generating $\mathrm{QCoh}\,\mathcal{X}$. Then $\mathrm{Lex}_R(\mathcal{C},A)$ is the essential image of the (fully faithful) functor $\Omega^*\colon\mathrm{QCoh}(\mathcal{X}_A)\to\mathrm{L}_R(\mathcal{C},A)$.*

If $\mathcal{X}$ is quasi-compact and all sheaves in $\mathcal{C}$ are finitely presented then $\mathrm{Lex}_R(\mathcal{C},A)$ is the subcategory of $\mathrm{L}_R(\mathcal{C},A)$ of functors which are exact on finite test sequences.

In particular, when $\mathcal{C}$ is essentially small,

$$\Omega^*\colon\mathrm{QCoh}(\mathcal{X}_A)\to\mathrm{Lex}_R(\mathcal{C},A) \text{ and } \mathcal{F}_{*,\mathcal{C}}\colon\mathrm{Lex}_R(\mathcal{C},A)\to\mathrm{QCoh}(\mathcal{X}_A)$$

are quasi-inverses of each other.

When $R=A=\mathbb{Z}$, $\mathcal{X}$ is a quasi-compact and quasi-separated scheme and $\mathcal{C}=\mathrm{Vect}(\mathcal{X})$, the category of locally free sheaves on $\mathcal{X}$, the results above have already been proved in [Bha16, Section 3.1]: the conclusion, which also extends in our setting, is that if $\mathcal{X}$ has the resolution property (that is $\mathrm{Vect}(\mathcal{X})$ generates $\mathrm{QCoh}(\mathcal{X})$) then $\mathrm{QCoh}\,\mathcal{X}$ can be recovered by $\mathrm{Vect}(\mathcal{X})$ and the distinguished class of morphisms which are surjective in the ambient category $\mathrm{QCoh}\,\mathcal{X}$.

When $\mathcal{C}$ is monoidal we obtain analogous statements for monoidal functors by replacing QCoh, L_R, Lex_R and $\mathcal{F}_{*,\mathcal{C}}$ with QAlg, ML_R, MLex_R and $\mathscr{A}_{*,\mathcal{C}}$ respectively (see 6.8).

Theorems A and B apply in the following situations in which $\mathcal{C}$ generates $\mathrm{QCoh}(\mathcal{X})$:

- if $\mathcal{C}=\mathrm{QCoh}\,\mathcal{X}$ then $\mathrm{Lex}_R(\mathrm{QCoh}\,\mathcal{X},A)$ is the category of contraviant, R-linear and left exact functors $\mathrm{QCoh}\,\mathcal{X}\to\mathrm{Mod}\,A$ which transform direct sums into products (see 6.2);
- if $\mathcal{C}=\mathrm{Coh}\,\mathcal{X}$ (resp. $\mathcal{C}=\mathrm{Vect}(\mathcal{X})$) and $\mathcal{X}$ is a Noetherian algebraic stack (resp. $\mathcal{X}$ is quasi-compact and has the resolution property) then $\mathrm{Lex}_R(\mathcal{C},A)$ is the category of contraviant, R-linear and left exact functors $\mathcal{C}\to\mathrm{Mod}\,A$ (see 6.5 and 6.4);
- if $\mathcal{C}=\mathrm{QCoh}_{\mathrm{fp}}\,\mathcal{X}$, the category of quasi-coherent sheaves of finite presentations, and $\mathcal{X}$ is a quasi-compact and quasi-separated scheme then $\mathrm{Lex}_R(\mathrm{QCoh}_{\mathrm{fp}}\,\mathcal{X},A)$ is the category of contraviant, R-linear functors $\mathrm{QCoh}_{\mathrm{fp}}\,\mathcal{X}\to\mathrm{Mod}\,A$ which are left exact on right exact sequences in $\mathrm{QCoh}_{\mathrm{fp}}\,\mathcal{X}$ (see 6.3).

When $\mathcal{C}$ is essentially small there is another cohomological characterization of the functors in $\mathrm{Lex}_R(\mathcal{C},A)$. A collection of maps $\mathcal{U}=\{\mathcal{E}_i\to\mathcal{E}\}$ in $\mathcal{C}$ is called jointly surjective if the map $\bigoplus_{i\in I}\mathcal{E}_i\to\mathcal{E}$ is surjective. Given such a collection $\mathcal{U}$ we set $\Delta_{\mathcal{U}}=\mathrm{Im}(\bigoplus_{i\in I}\Omega^{\mathcal{E}_i}\to\Omega^{\mathcal{E}})\in\mathrm{L}_R(\mathcal{C},R)$. Denote by $\mathcal{C}^{\oplus}$ the subcategory of $\mathrm{QCoh}\,\mathcal{X}$ consisting of all possible finite direct sums of sheaves in $\mathcal{C}$. We have:

Theorem C (5.5, 5.7 and 5.9). *Let $\mathcal{X}$ be a pseudo-algebraic category fibered in groupoids over R and $\mathcal{C} \subseteq \operatorname{QCoh} \mathcal{X}$ be a full and essentially small subcategory generating $\operatorname{QCoh} \mathcal{X}$. Then $\operatorname{Lex}_R(\mathcal{C}, A)$ is the full subcategory of $\mathrm{L}_R(\mathcal{C}, A)$ of functors Γ satisfying*

$$\operatorname{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\Omega^{\mathcal{E}}/\Delta_{\mathcal{U}}, \Gamma) = \operatorname{Ext}^1_{\mathrm{L}_R(\mathcal{C},R)}(\Omega^{\mathcal{E}}/\Delta_{\mathcal{U}}, \Gamma) = 0$$

for all jointly surjective collections of maps $\mathcal{U} = \{\mathcal{E}_i \to \mathcal{E}\}_{i \in I}$ in $\mathcal{C}$. If $\mathcal{X}$ is quasi-compact and the sheaves in $\mathcal{C}$ are finitely presented we can consider only finite collections $\mathcal{U}$.

We have $\operatorname{Lex}_R(\mathcal{C}^{\oplus}, A) \simeq \operatorname{Lex}_R(\mathcal{C}, A)$ via the restriction $\mathcal{C} \to \mathcal{C}^{\oplus}$ and, if $\mathcal{C}$ is additive and $\mathcal{J}$ is the smallest Grothendieck topology on $\mathcal{C}$ containing the sieves $\Delta_{\mathcal{U}}$ for all jointly surjective collections $\mathcal{U} = \{\mathcal{E}_i \to \mathcal{E}\}_{i \in I}$ in $\mathcal{C}$, then $\operatorname{Lex}_R(\mathcal{C}, A)$ coincides with the category of sheaves of A-modules $\mathcal{C}^{\mathrm{op}} \to \operatorname{Mod} A$ on the site $(\mathcal{C}, \mathcal{J})$ which are R-linear.

Theory above find applications in the following situations.

- In [Sch20] the author proves that bicategorical limits and colimits exist in the 2-category of Adam stacks, that is of quasi-compact stacks with affine diagonal and the resolution property. The class of Adam stacks is therefore a cocomplete class of "geometric" objects, whilst the class of Artin stack is not cocomplete.
- In [Ton20], given a quasi-compact fibered category $\mathcal{X}$ and a monoidal subcategory $\mathcal{C}$ of $\operatorname{Vect}(\mathcal{X})$ I study the stack $\operatorname{Fib}_{\mathcal{X},\mathcal{C}}$ of fiber functors with source $\mathcal{C}$, which are particular right exact and strong monoidal functors. This stack comes equipped with functors $\mathcal{P}_{\mathcal{C}} \colon \mathcal{X} \to \operatorname{Fib}_{\mathcal{X},\mathcal{C}}$ and $\mathcal{G} \colon \mathcal{C} \to \operatorname{Vect}(\operatorname{Fib}_{\mathcal{X},\mathcal{C}})$ and, if $\mathcal{C}$ generates $\operatorname{QCoh}(\mathcal{X})$, Theorems A and B are used to show that $\mathcal{X} \to \operatorname{Fib}_{\mathcal{X},\mathcal{C}}$ is an equivalence. In general it is proved that $\operatorname{Fib}_{\mathcal{X},\mathcal{C}}$ is a quasi-compact stack and $\mathcal{G}(\mathcal{C})$ generates $\operatorname{QCoh}(\operatorname{Fib}_{\mathcal{X},\mathcal{C}})$ under mild hypothesis on $\mathcal{C}$.
- In [Ton17] it is applied to the theory of Galois covers. As explained in the beginning of this introduction, in my Ph.D. thesis [Ton13] I have worked out theory above in the case $\mathcal{X} = \mathrm{B}\, G$ and $\mathcal{C} = \operatorname{Vect} \mathcal{X}$, where G is a finite, flat and finitely presented group scheme over R ($\mathcal{C}$ generates $\operatorname{QCoh}(\mathrm{B}\, G)$ is this case, see 8.3). The proof presented in [Ton13] makes use of representation theory and can not be generalized to arbitrary categories fibered in groupoids. Moreover the results in the present paper also apply to more general affine group schemes over R, for instance any flat affine group scheme defined over a Dedekind domain (see 8.3 and 8.6). The goal of [Ton13] and [Ton17] was to look at Galois covers with group G as particular monoidal functors, as G-torsors can be thought of as particular strong monoidal functors, and the motivation was the study of non-abelian Galois covers, where a direct approach as in the abelian case (see [Ton14]) fails due to the complexity of the representation theory.
- In [DTZ] we discuss a generalization of Drinfeld's lemma. Let $\mathcal{X}$ be a fibered category over a finite field $\mathbb{F}_q$, $\mathbb{F}_q \subseteq k$ an algebraically closed field, $\phi_k \colon k \to k$, $\phi_k(x) = x^q$ the Frobenius, $\mathcal{X}_k = \mathcal{X} \times_{\mathbb{F}_q} k$ and $\phi_k = \mathrm{id} \times \phi_k \colon \mathcal{X}_k \to \mathcal{X}_k$ the geometric Frobenius. Then there is a natural functor
$$\Psi_{\mathcal{X}} \colon \operatorname{QCoh}_{\mathrm{fp}}(\mathcal{X}) \to \operatorname{QCoh}_{\mathrm{fp}}(\mathcal{X}, \phi_k)$$
where $\operatorname{QCoh}_{\mathrm{fp}}(\mathcal{X})$ is the category of finitely presented quasi-coherent sheaves, while $\operatorname{QCoh}_{\mathrm{fp}}(\mathcal{X}, \phi_k)$ is the category of pairs $(\mathcal{F}, \sigma)$ where $\mathcal{F} \in \operatorname{QCoh}_{\mathrm{fp}}(\mathcal{X}_k)$ and $\sigma \colon \phi_k^* \mathcal{F} \to \mathcal{F}$ is an isomorphism. Drinfeld proved that the functor $\Psi_{\mathcal{X}}$ is an equivalence when $\mathcal{X}$ is a projective scheme over $\mathbb{F}_q$ and its proof used the classical correspondence between quasi-coherent sheaves on a projective space and graded modules. Using the generalized correspondence between sheaves and linear functors, it is possible to extend Drinfeld argument to more general schemes and stacks, for instance proper algebraic stacks or arbitrary affine gerbes over $\mathbb{F}_q$.

- Another application, which will be hopefully the subject of a future paper, is to the theory of Cox rings and homogeneous sheafifications. This is a generalization of the results on quasi-projective schemes described in Section 7. The idea is to consider $\mathcal{C}_H = \{\mathcal{L}\}_{\mathcal{L}\in H} \subseteq \operatorname{Vect}\mathcal{X}$ where H is a subgroup of $\operatorname{Pic}\mathcal{X}$. As in the projective case we have a homogeneous coordinate ring
$$S_H = \bigoplus_{\mathcal{L}\in H} \mathrm{H}^0(\mathcal{X},\mathcal{L})$$
(opportunely defined, see [HMT20, Theorem A]), $\mathrm{L}_R(\mathcal{C}_H, A)$ is equivalent to $\operatorname{GMod}(S_H\otimes_R A)$, the category of H-graded $(S_H\otimes_R A)$-modules, Ω^* corresponds to
$$\Gamma_*\colon \operatorname{QCoh}(\mathcal{X}_A)\to \operatorname{GMod}(S_H\otimes_R A),\ \Gamma_*(\mathcal{F}) = \bigoplus_{\mathcal{L}\in H} \mathrm{H}^0(\mathcal{X},\mathcal{F}\otimes\mathcal{L})$$
and its adjoint $\mathcal{F}_{*,\mathcal{C}_H}$ behaves like a homogeneous sheafification. Moreover in more concrete geometric situations, e.g. when $\mathcal{X}$ is a normal variety, there are analogous constructions for reflexive sheaves of rank 1 (see [HMT20, Theorem B]). We expect that this theory covers all known cases where Γ_* is proved to be fully faithful (see for instance [CLS11, Appendix of Chapter 6] and [Kaj98, Section 2]).

The paper is divided as follows. In the first section we explain how to work with quasi-coherent sheaves on fibered categories, while in the second one we introduce sheafification functors and prove basic properties. In the third section we discuss when the Yoneda functor Ω^* is fully faithful, while in the fourth and fifth ones we determine its essential image. The last sections are about applications.

Notation

In this paper we work over a base commutative, associative ring R with unity. If not stated otherwise a fiber category will be a category fibered in groupoids over Aff/R, the category of affine schemes over $\operatorname{Spec} R$, or, equivalently, the opposite of the category of R-algebras. Recall that by the 2-Yoneda lemma objects of a fibered category $\mathcal{X}$ can be thought of as maps $T\to\mathcal{X}$ from an affine scheme. An fpqc stack will be a stack for the fpqc topology.

A map $f\colon \mathcal{X}'\to\mathcal{X}$ of fibered categories is called representable if for all maps $T\to\mathcal{X}$ from an affine scheme (or an algebraic space) the fiber product $T\times_{\mathcal{X}}\mathcal{X}'$ is (equivalent to) an algebraic space.

Given a flat and affine group scheme G over R we denote by $\mathrm{B}_R G$ the stack of G-torsors for the *fpqc* topology, which is an fpqc stack with affine diagonal. When $G\to\operatorname{Spec} R$ is finitely presented (resp. smooth) then $\mathrm{B}_R G$ coincides with the stack of G-torsors for the fppf (resp. étale) topology.

By a "subcategory" of a given category we mean a "full subcategory" if not stated otherwise.

We are going to look at functors whose source category $\mathcal{C}$ will be a full subcategory of quasi-coherent sheaves of some fibered category. Many construction makes sense only when $\mathcal{C}$ is small, yet the results obtained are clearly true also for an essentially small category $\mathcal{C}$. Therefore in the text we will only consider small categories, but keeping in mind the previous observation.

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1. Preliminaries on sheaves and fibered categories

Many definition and properties showed in this section can be also found in [TV18, Section 4.1]. The notion of modules and quasi-coherent sheaves makes sense on an arbitrary ringed site.

Definition 1.1. Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be a ringed category. A presheaf of $\mathcal{O}_{\mathcal{C}}$-modules if a contravariant functor $\mathcal{F}\colon \mathcal{C} \to (\mathrm{Ab})$ endowed, for any $T \in \mathcal{C}$, of an $\mathcal{O}_{\mathcal{C}}(T)$-module structure on $\mathcal{F}(T)$ such that, for any $T' \to T$ in $\mathcal{C}$ the map $\mathcal{F}(T) \to \mathcal{F}(T')$ is $\mathcal{O}_{\mathcal{C}}(T)$-linear. We denote by $\mathrm{Mod}(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ or simply $\mathrm{Mod}\, \mathcal{O}_{\mathcal{C}}$ the category of presheaves of $\mathcal{O}_{\mathcal{C}}$-modules.

Assume now $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is a ringed site. A quasi-coherent sheaf on $\mathcal{C}$ is a sheaf of $\mathcal{O}_{\mathcal{C}}$-modules such that, for any $T \in \mathcal{C}$, there exist a covering $\{T_i \to T\}$ and an exact sequence of sheaves on the restricted site $\mathcal{C}/T_i$

$$(\mathcal{O}_{\mathcal{C}})^{(I)}_{|T_i} \to (\mathcal{O}_{\mathcal{C}})^{(J)}_{|T_i} \to \mathcal{F}_{|T_i} \to 0$$

where I and J are sets and $\mathcal{G}_{|T_i} = \mathcal{G} \circ r_{T_i}$, $r_{T_i}\colon \mathcal{C}/T_i \to \mathcal{C}$ and $\mathcal{G} \in \mathrm{Mod}\, \mathcal{O}_{\mathcal{C}}$. We denote by $\mathrm{QCoh}\, \mathcal{C}$ the category of quasi-coherent sheaves on $\mathcal{C}$.

Let $\pi\colon \mathcal{X} \to \mathrm{Aff}/R$ be a fibered category. There is a functor of rings $\mathcal{O}_{\mathcal{X}}\colon \mathcal{X}^{\mathrm{op}} \to (\mathrm{Sets})$ defined by $\mathcal{O}_{\mathcal{X}}(\xi) = \mathrm{H}^0(\mathcal{O}_{\pi(\xi)})$, so that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a ringed category.

Definition 1.2. A quasi-coherent sheaf over $\mathcal{X}$ is a presheaf of $\mathcal{O}_{\mathcal{X}}$-modules such that for all maps $\xi \to \eta$ in $\mathcal{X}$ the induced map

$$\mathcal{F}(\eta) \otimes_{\mathrm{H}^0(\mathcal{O}_{\pi(\eta)})} \mathrm{H}^0(\mathcal{O}_{\pi(\xi)}) \to \mathcal{F}(\xi)$$

is an isomorphism. We denote by $\mathrm{QCoh}\, \mathcal{X}$ the full subcategory of $\mathrm{Mod}\, \mathcal{O}_{\mathcal{X}}$ of quasi-coherent sheaves.

If $\mathscr{C}$ is a (not full) subcategory of $\mathcal{X}$ we similarly define presheaves of $(\mathcal{O}_{\mathcal{X}})_{|\mathscr{C}}$-modules and quasi-coherent sheaves on $\mathscr{C}$ just replacing all occurrences of $\mathcal{X}$ with $\mathscr{C}$. We denote by $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})_{|\mathscr{C}}$ and $\mathrm{QCoh}\, \mathscr{C}$ the resulting categories.

Proposition 1.3. *Let $\mathcal{X} \to \mathrm{Aff}/R$ be a fibered category and $\mathcal{F} \in \mathrm{Mod}\, \mathcal{O}_{\mathcal{X}}$. Then $\mathcal{F}$ is quasi-coherent if and only if it is quasi-coherent on the site $\mathcal{X}$ with the Zariski (étale, fppf, fpqc) topology. In particular quasi-coherent sheaves are sheaves in the fpqc topology.*

Proof. Let us denote by $\mathcal{X}_Z$, $\mathcal{X}_E$, $\mathcal{X}_F$, $\mathcal{X}_{\overline{F}}$ the category $\mathcal{X}$ with the Zariski, étale, fppf, fpqc topology respectively. Inside $\mathrm{Mod}\, \mathcal{O}_{\mathcal{X}}$ there are inclusions

$$\mathrm{QCoh}\, \mathcal{X}_Z \subseteq \mathrm{QCoh}\, \mathcal{X}_E \subseteq \mathrm{QCoh}\, \mathcal{X}_F \subseteq \mathrm{QCoh}\, \mathcal{X}_{\overline{F}}$$

Moreover it is clear that we can assume that $\mathcal{X}$ is an affine scheme, in which case the result has been proved in [TV18, Prop 4.13]. □

If $f\colon \mathcal{Y} \to \mathcal{X}$ is a morphism of fibered categories and $\mathcal{F} \in \mathrm{Mod}\, \mathcal{O}_{\mathcal{X}}$ we define $f^*\mathcal{F} = \mathcal{F} \circ f\colon \mathcal{Y}^{\mathrm{op}} \to \mathcal{X}^{\mathrm{op}} \to (\mathrm{Sets})$. This association defines a functor $f^*\colon \mathrm{Mod}\, \mathcal{O}_{\mathcal{X}} \to \mathrm{Mod}\, \mathcal{O}_{\mathcal{Y}}$, called the pull-back functor, and restricts to a functor $f^*\colon \mathrm{QCoh}\, \mathcal{X} \to \mathrm{QCoh}\, \mathcal{Y}$. Notice that $f^*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$ tautologically.

The category $\mathrm{Mod}\, \mathcal{O}_{\mathcal{X}}$ is an abelian category: if $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map and $\xi\colon \mathrm{Spec}\, B \to \mathcal{X}$ an object then

$$\widetilde{\mathrm{Ker}}(\alpha)(\xi) = \mathrm{Ker}(\alpha_\xi\colon \mathcal{F}(\xi) \to \mathcal{G}(\xi)),\ \widetilde{\mathrm{Im}}(\alpha)(\xi) = \mathrm{Im}(\alpha_\xi\colon \mathcal{F}(\xi) \to \mathcal{G}(\xi))$$

$$\mathrm{Coker}(\alpha)(\xi) = \mathrm{Coker}(\alpha_\xi\colon \mathcal{F}(\xi) \to \mathcal{G}(\xi))$$

are the kernel, image and cokernel of α in $\mathrm{Mod}\, \mathcal{O}_{\mathcal{X}}$. If $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent then so is $\mathrm{Coker}(\alpha)$ because our category $\mathcal{X}$ is fibered over affine schemes. In particular $\alpha\colon \mathcal{F} \to \mathcal{G}$ is an epimorphism if and only if it is surjective objectwise. On the other hand kernels and images are almost never quasi-coherent, essentially because pull-backs are not left exact. The category $\mathrm{QCoh}\,\mathcal{X}$ is R-linear but it is unclear if it is abelian. There is a natural condition on $\mathcal{X}$ which allows us to prove that $\mathrm{QCoh}\,\mathcal{X}$ is an R-linear abelian category.

Definition 1.4. An *fpqc atlas* (or simply *atlas*) of a fibered category $\mathcal{X}$ is a representable fpqc covering $X \to \mathcal{X}$ from a scheme. A fiber category is called *pseudo-algebraic* if it has an atlas, it is called *quasi-compact* if it has an atlas from an affine scheme.

Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of fibered categories. The map f is called *pseudo-algebraic* (resp. *quasi-compact*) if for all maps $T \to \mathcal{X}$ from a scheme (resp. quasi-compact scheme) the fiber product $T \times_{\mathcal{X}} \mathcal{Y}$ is pseudo-algebraic (resp. quasi-compact). It is called *quasi-separated* if the diagonal $\mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is quasi-compact.

Remark 1.5. Notice that in [TV18, Definition 3.18] an atlas $X \to \mathcal{X}$ is required to be schematically representable. In this paper we don't need this assumption. Clearly if $\mathcal{X}$ has schematically representable diagonal the two notions agree.

Remark 1.6. A map of fibered categories which is locally (in some topology) representable is not in general representable: if $F \to S$ is a map of functors, $T \to S$ is a covering in some topology and $F \times_S T$ is an algebraic space we cannot conclude that F is a sheaf in the same topology. On the other hand a map of stacks in the étale topology which is fppf locally representable it is representable: this is because a sheaf in the étale topology which is fppf local an algebraic space is an algebraic space. It is not clear if a map of fpqc stacks which is fpqc locally (schematically) representable is representable. Again the problem is the following: if an fpqc sheaf is fpqc locally on the base an algebraic space (or even a scheme), then is it an algebraic space itself?

In our situation we can see that if $\mathcal{X}$ is pseudo-algebraic fibered category then the diagonal $\Delta_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X} \times_R \mathcal{X}$ is only fpqc locally representable.

Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a map of fibered categories. If $\mathcal{X}$ and f are pseudo-algebraic then $\mathcal{Y}$ is pseudo-algebraic. If $\mathcal{Y}$ is pseudo-algebraic and $\Delta_{\mathcal{X}}$ is representable then f is pseudo-algebraic.

Definition 1.7. We define $\mathcal{X}_{\mathrm{fl}}$ (resp. $\mathcal{X}_{\mathrm{sm}}$, $\mathcal{X}_{\mathrm{et}}$) as the (not full) subcategory of $\mathcal{X}$ of objects $\xi\colon \operatorname{Spec} B \to \mathcal{X}$ which are representable and flat (resp. smooth, étale) and the arrows are morphisms in $\mathcal{X}$ whose underlying map of affine schemes is flat (resp. smooth, étale).

If $X \to \mathcal{X}$ is an fpqc atlas then by definition $V = X \times_{\mathcal{X}} X$ is an algebraic space and the two projections $V \rightrightarrows X$ extends to a groupoid in algebraic spaces. We denote by $\mathrm{QCoh}(V \rightrightarrows X)$ the category of quasi-coherent sheaves on $V \rightrightarrows X$ (see [Aut19, Tag 0440]). By standard arguments of fpqc descent for modules we have (see also [TV18, Proposition 4.6]):

Proposition 1.8. *If $\mathcal{X}$ admits an fpqc (resp. smooth, étale) atlas then the restriction $\mathrm{QCoh}\,\mathcal{X} \to \mathrm{QCoh}\,\mathcal{X}_{fl}$ (resp. $\mathrm{QCoh}\,\mathcal{X}_{sm}$, $\mathrm{QCoh}\,\mathcal{X}_{et}$) is an equivalence of categories. If $f\colon X \to \mathcal{X}$ is an fpqc atlas then $f^*\colon \mathrm{QCoh}\,\mathcal{X} \to \mathrm{QCoh}\,X$ is faithful and it induces an equivalence $\mathrm{QCoh}\,\mathcal{X} \to \mathrm{QCoh}(V \rightrightarrows X)$.*

We see that if $\mathcal{X}$ is pseudo-algebraic then $\mathrm{QCoh}\,\mathcal{X}$ is equivalent to an R-linear abelian category, namely $\mathrm{QCoh}(R \rightrightarrows X)$. Moreover if $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent sheaves then $\mathrm{Ker}(\alpha)$ is defined by taking $\mathrm{Ker}(\alpha_{|\mathcal{X}_{\mathrm{fl}}}) \in \mathrm{QCoh}\,\mathcal{X}_{\mathrm{fl}}$, which is just given by $\mathrm{Ker}(\alpha_{|\mathcal{X}_{\mathrm{fl}}})(\operatorname{Spec} B \to \mathcal{X}) = \mathrm{Ker}(\alpha(\xi)\colon \mathcal{F}(\xi) \to \mathcal{G}(\xi))$ for $\xi \in \mathcal{X}_{\mathrm{fl}}$, and then extending it to the whole $\mathcal{X}$. If $\mathcal{X}$ is an algebraic stack or a scheme we see that $\mathrm{QCoh}\,\mathcal{X}$ is equivalent to the usual category of quasi-coherent sheaves via an R-linear and exact functor.

Proposition 1.9. *If $\mathcal{X}$ is pseudo-algebraic then a sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of quasi-coherent sheaves is exact in $\mathrm{QCoh}(\mathcal{X})$ if and only if it is exact in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$. In particular if $f\colon \mathcal{Y} \to \mathcal{X}$ is a map from a pseudo-algebraic fibered category then $f^*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is right exact on right exact sequences.*

Proof. Via the equivalence $\mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{X}_{\mathrm{fl}})$ the sequence $\mathcal{S}$ is the statement is exact in $\mathrm{QCoh}(\mathcal{X})$ if and only if $\mathcal{S}(B)$ is exact for all $\operatorname{Spec} B \to \mathcal{X}$ flat and representable. So if $\mathcal{S}$ is exact in $\mathrm{Mod}(\mathcal{X})$ then it is exact in $\mathrm{QCoh}(\mathcal{X})$. Conversely if $X \to \mathcal{X}$ is an atlas and $\operatorname{Spec} B \to \mathcal{X}$ is any map then $\mathcal{S}(B)$ is exact because it is so after the fpqc covering $X \times_{\mathcal{X}} B \to \operatorname{Spec} B$. □

Definition 1.10. Given a subcategory $\mathcal{D}$ of $\mathrm{QCoh}\,\mathcal{X}$, by an exact sequence of sheaves in $\mathcal{D}$ will always mean exact sequence in $\mathrm{QCoh}\,\mathcal{X}$ of sheaves belonging to $\mathcal{D}$.

We now deal with the problem of defining a right adjoint of a pull-back functor, that is a push-forward. Given $\mathcal{F} \in \mathrm{Mod}\,\mathcal{O}_{\mathcal{X}}$ we define the global section $\mathcal{F}(\mathcal{X}) = \mathrm{Hom}(\mathcal{O}_{\mathcal{X}}, \mathcal{F})$ of $\mathcal{F}$, also denoted by $\mathrm{H}^0(\mathcal{X}, \mathcal{F})$ or simply $\mathrm{H}^0(\mathcal{F})$, which is an $\mathcal{O}_{\mathcal{X}}(\mathcal{X})$-module. More generally given a map of fibered categories $g\colon \mathcal{Z} \to \mathcal{X}$ we define $\mathcal{F}(\mathcal{Z} \to \mathcal{X}) = (g^*\mathcal{F})(\mathcal{Z})$, sometimes simply written as $\mathcal{F}(\mathcal{Z})$. If $\mathcal{Z} = \operatorname{Spec} B$ is affine we will often write $\mathcal{F}(B)$ instead of $\mathcal{F}(\operatorname{Spec} B)$.

Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a map of fibered categories. The Weil restriction defines a functor $f_p\colon \mathrm{Mod}\,\mathcal{O}_{\mathcal{Y}} \to \mathrm{Mod}\,\mathcal{O}_{\mathcal{X}}$: given $\mathcal{H} \in \mathrm{Mod}\,\mathcal{O}_{\mathcal{Y}}$ and an object $\xi\colon T \to \mathcal{X}$ of $\mathcal{X}$ we define

$$(f_p\mathcal{H})(\xi) = \mathcal{H}(T \times_{\mathcal{X}} \mathcal{Y})$$

Proposition 1.11. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a map of fibered categories. Then f_p is a right adjoint of f^* and, if*

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{g'} & \mathcal{Y} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{X}' & \xrightarrow{g} & \mathcal{X} \end{array}$$

is a 2-cartesian diagram of fibered categories, there is an isomorphism of functors

$$g^* f_p \to f'_p g'^*\colon\ \mathrm{Mod}\,\mathcal{O}_{\mathcal{Y}} \to \mathrm{Mod}\,\mathcal{O}_{\mathcal{X}'}$$

If f is affine then $f_p(\mathrm{QCoh}\,\mathcal{Y}) \subseteq \mathrm{QCoh}\,\mathcal{X}$ and $(f_p)_{|\,\mathrm{QCoh}\,\mathcal{Y}}\colon \mathrm{QCoh}\,\mathcal{Y} \to \mathrm{QCoh}\,\mathcal{X}$ is right adjoint to $f^*\colon \mathrm{QCoh}\,\mathcal{X} \to \mathrm{QCoh}\,\mathcal{Y}$.

Proof. We define $f^* f_p\mathcal{H} \to \mathcal{H}$ for $\mathcal{H} \in \mathrm{Mod}\,\mathcal{O}_{\mathcal{Y}}$ as

$$f^*(f_p\mathcal{H})(T \to \mathcal{Y}) = (f_p\mathcal{H})(T \to \mathcal{Y} \to \mathcal{X}) = \mathcal{H}(\mathcal{Y} \times_{\mathcal{X}} T \to \mathcal{Y}) \to \mathcal{H}(T \to \mathcal{Y})$$

Conversely we define $\mathcal{G} \to f_p f^*\mathcal{G}$ for $\mathcal{G} \in \mathrm{Mod}\,\mathcal{O}_{\mathcal{X}}$ as

$$\mathcal{G}(T \to \mathcal{X}) \to \mathcal{G}(\mathcal{Y} \times_{\mathcal{X}} T \to T \to \mathcal{X}) = f^*\mathcal{G}(\mathcal{Y} \times_{\mathcal{X}} T \to \mathcal{Y}) = f_p(f^*\mathcal{G})(T \to \mathcal{X})$$

where the last map is induced by the given section $T \to T \times_{\mathcal{X}} \mathcal{Y}$. We have to prove that the corresponding maps

$$\Psi\colon\ \mathrm{Hom}(f^*\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}(\mathcal{G}, f_p\mathcal{H}),\ \ \Phi\colon\ \mathrm{Hom}(\mathcal{G}, f_p\mathcal{H}) \to \mathrm{Hom}(f^*\mathcal{G}, \mathcal{H})$$

are inverses of each other. Given a map $T \to \mathcal{X}$ we set $\mathcal{Y}_T = \mathcal{Y} \times_{\mathcal{X}} T$. Given $\sigma\colon f^*\mathcal{G} \to \mathcal{H}$ and $T \to \mathcal{Y}$ we have commutative diagrams

$$\begin{array}{ccccc} \mathcal{G}(T \to \mathcal{X}) & \longrightarrow & \mathcal{G}(\mathcal{Y}_T \to \mathcal{X}) & & \\ \| & & \| & & \\ f^*\mathcal{G}(T \to \mathcal{Y}) & \xrightarrow{a} & f^*\mathcal{G}(\mathcal{Y}_T \to \mathcal{Y}) & \xrightarrow{\sigma_{\mathcal{Y}_T}} & \mathcal{H}(\mathcal{Y}_T \to \mathcal{Y}) \\ & \searrow^{\mathrm{id}} & \downarrow & & \downarrow{\scriptstyle c} \\ & & f^*\mathcal{G}(T \to \mathcal{Y}) & \xrightarrow{\sigma_T} & \mathcal{H}(T \to \mathcal{Y}) \end{array}$$

The vertical maps are induced by the section $T \to \mathcal{Y}_T$, while a by the projection $\mathcal{Y}_T \to T$. The composition $c\sigma_{\mathcal{Y}_T} a$ equals $\Phi\Psi(\sigma)_T$ and, thanks to the above diagram, $\Phi\Psi(\sigma)_T = \sigma$. Conversely let $\delta\colon \mathcal{G} \to f_p\mathcal{H}$ and $T \to \mathcal{X}$. We have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{G}(T) & \xrightarrow{\delta_T} & f_p\mathcal{H}(T) & & \\
\downarrow{\scriptstyle a} & & \downarrow & \overset{\mathrm{id}}{\searrow} & \\
\mathcal{G}(\mathcal{Y}_T) & \xrightarrow{\delta_{\mathcal{Y}_T}} & f_p\mathcal{H}(\mathcal{Y}_T) & \xrightarrow{c} & f_p\mathcal{H}(T)
\end{array}$$

The vertical maps are induced by $\mathcal{Y}_T \to T$, while c by the diagonal $\mathcal{Y}_T \to \mathcal{Y}_T \times_{\mathcal{X}} \mathcal{Y}_T$. One can check that $c\delta_{\mathcal{Y}_T} a$ equals $\Psi\Phi(\delta)_T$ and, thanks to the above diagram, $\psi\Phi(\delta)_T = \delta_T$.

The isomorphism for the base change is tautological. For the last claim we can assume that $\mathcal{X}$ is an affine scheme in which case the result follows because (usual) push-forwards commutes with arbitrary base changes. $\square$

In general f_p does not preserve quasi-coherent sheaves, even if f is a proper map of schemes. To get a right adjoint of pullback we have to require more.

Definition 1.12. A pseudo-algebraic map $f\colon \mathcal{Y} \to \mathcal{X}$ of fibered categories is called *flat* if given an object $\xi\colon T \to \mathcal{X}$ of $\mathcal{X}$ and an atlas $V \to T \times_{\mathcal{X}} \mathcal{Y}$ the resulting map $V \to T$ is flat.

If $f\colon \mathcal{Y} \to \mathcal{X}$ is a map of algebraic stacks then the above notion extends the classical one, which is made only using smooth atlases. Indeed one reduces easily to the case of schemes, in which case by hypothesis there is an fpqc covering $g\colon \mathcal{Z} \to \mathcal{X}$ such that $\mathcal{Z} \to \mathcal{X}$ is flat. It follows easily that $\mathcal{Y} \to \mathcal{X}$ is also flat.

Proposition 1.13. *If $f\colon \mathcal{Y} \to \mathcal{X}$ is a flat map of pseudo-algebraic fibered categories then $f^*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact.*

Proof. By 1.9 we have to show that f^* maps a monomorphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{QCoh}(\mathcal{X})$ to a monomorphism in $\mathrm{QCoh}(\mathcal{Y})$. From 1.8, if $\xi\colon \operatorname{Spec} B \to \mathcal{Y}$ is a representable and flat map we have to show that $\xi^* f^*\mathcal{F} \to \xi^* f^*\mathcal{G}$ is injective. So we can assume $\mathcal{Y} = \operatorname{Spec} B$. If $\pi\colon X \to \mathcal{X}$ is an atlas consider the Cartesian diagram

$$\begin{array}{ccc}
Y & \xrightarrow{h} & \operatorname{Spec} B \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\pi} & \mathcal{X}
\end{array}$$

By hypothesis Y is an algebraic space, $g\colon Y \to X$ is flat and $h\colon Y \to \operatorname{Spec} B$ is an fpqc covering. Moreover $\pi^*\mathcal{F} \to \pi^*\mathcal{G}$ is injective by 1.8. Thus $h^* f^*\mathcal{F} \to h^* f^*\mathcal{G}$ is injective and therefore $f^*\mathcal{F} \to f^*\mathcal{G}$ is injective as well. $\square$

Proposition 1.14. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a map from a pseudo-algebraic stack to a quasi-separated scheme and such that $f^*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact. Then f is flat.*

Proof. We first reduce to the case when $\mathcal{X}$ is affine. Let $U \subseteq \mathcal{X}$ be an affine open subset. The inclusion map $i\colon U \to \mathcal{X}$ is quasi-compact since $\mathcal{X}$ is quasi-separated and quasi-separated because a monomorphism. In particular $i_*\colon \mathrm{QCoh}(U) \to \mathrm{QCoh}(\mathcal{X})$ is well defined. Consider the Cartesian diagram

$$\begin{array}{ccc}
Y & \xrightarrow{j} & \mathcal{Y} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
U & \xrightarrow{i} & \mathcal{X}
\end{array}$$

Given an injective map $\mathcal{G} \to \mathcal{F}$ in $\mathrm{QCoh}(U)$ we have to show that $g^*(\mathcal{G} \to \mathcal{F})$ is still injective. Consider the exact sequence $\mathcal{H}_* \colon 0 \to \mathcal{K} \to i_*\mathcal{G} \to i_*\mathcal{F}$. Since $i^*i_* \simeq \mathrm{id}$ it follows that

$$j^*f^*\mathcal{H}_* \simeq g^*i^*\mathcal{H}_* \simeq g^*(0 \to 0 \to \mathcal{G} \to \mathcal{F})$$

Since f^* is exact and j is flat the above sequence is exact as required.

So we can assume $\mathcal{X}$ affine. if $V \to \mathcal{Y}$ is an fpqc atlas from a scheme then $V \to \mathcal{X}$ is flat: if $W \subseteq V$ is an open affine subset then $\mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(V)$ is exact and therefore $W \to \mathcal{X}$ is flat. Since for any $T \to \mathcal{X}$ the map $V \times_{\mathcal{X}} T \to \mathcal{Y} \times_{\mathcal{X}} T$ is an fpqc atlas, by definition of flatness we can conclude that $\mathcal{Y} \to \mathcal{X}$ is flat. □

Proposition 1.15. *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a quasi-compact and quasi-separated map of pseudo-algebraic fibered categories. Then the composition $\mathrm{QCoh}\,\mathcal{Y} \to \mathrm{Mod}\,\mathcal{O}_{\mathcal{X}} \to \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})_{|\mathcal{X}_{\mathrm{fl}}}$ has values in $\mathrm{QCoh}\,\mathcal{X}_{\mathrm{fl}}$. The induced map $f_* \colon \mathrm{QCoh}\,\mathcal{Y} \to \mathrm{QCoh}\,\mathcal{X}$ is a right adjoint of $f^* \colon \mathrm{QCoh}\,\mathcal{X} \to \mathrm{QCoh}\,\mathcal{Y}$. If*

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{g'} & \mathcal{Y} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{X}' & \xrightarrow{g} & \mathcal{X} \end{array}$$

is a 2-cartesian diagram of fibered categories with $\mathcal{X}'$ pseudo-algebraic then $\mathcal{Y}'$ is pseudo-algebraic, f' is quasi-compact and quasi-separated and there is a natural transformation of functors

$$g^*f_* \to f'_*g'^* \colon \mathrm{QCoh}\,\mathcal{X} \to \mathrm{QCoh}\,\mathcal{Y}'$$

which is an isomorphism if g is flat.

Proof. Consider the 2-Cartesian diagram in the statement. The diagonal of f' is quasi-compact because it is base change of the diagonal of f. To see that $f_p(\mathcal{F})_{|\mathcal{X}_{\mathrm{fl}}}$ is quasi-coherent for $\mathcal{F} \in \mathrm{QCoh}\,\mathcal{Y}$, we can assume $\mathcal{X} = \mathrm{Spec}\,B$ affine and that $\mathcal{Y}$ is quasi-compact with quasi-compact diagonal. If $U = \mathrm{Spec}\,A \to \mathcal{Y}$ is a fpqc atlas, it follows that $R = U \times_{\mathcal{Y}} U$ is a quasi-compact algebraic space. By covering R by finitely many affine schemes $\mathrm{Spec}\,A_i$ we can write $\mathcal{F}(\mathcal{Y})$ as kernel of a map $\mathcal{F}(A) \to \oplus_i \mathcal{F}(A_i)$. If we base change along a flat map $B \to B'$ it is now easy to see that $\mathcal{F}(\mathcal{Y} \times_B B') \simeq \mathcal{F}(\mathcal{Y}) \otimes_B B'$, as required.

To define the natural transformation $\alpha \colon g^*f_* \to f'_*g'^*$ notice that there is a natural map $f_*\mathcal{F} \to f_p\mathcal{F}$ which extends the identity on $\mathcal{X}_{\mathrm{fl}}$. Applying g^* we get $g^*f_*\mathcal{F} \to g^*f_p\mathcal{F} \simeq f'_p g'^*\mathcal{F}$ and then, restricting to $\mathcal{X}'_{\mathrm{fl}}$, a map $(g^*f_*\mathcal{F})_{|\mathcal{X}'_{\mathrm{fl}}} \to (f'_*g'^*\mathcal{F})_{|\mathcal{X}'_{\mathrm{fl}}}$. Since both sides are in $\mathrm{QCoh}\,\mathcal{X}'_{\mathrm{fl}}$ this map uniquely extends to a natural transformation α as required. Finally assume that g is flat and let $\xi \colon \mathrm{Spec}\,B \to \mathcal{X}' \in \mathcal{X}'_{\mathrm{fl}}$. If the composition $\mathrm{Spec}\,B \to \mathcal{X}$ is in $\mathcal{X}_{\mathrm{fl}}$ then one can easily check that $\alpha(\xi)$ is an isomorphism. Otherwise, by definition of flatness, there exists an fpqc covering $\mathrm{Spec}\,B' \to \mathrm{Spec}\,B$ whose composition $\xi' \colon \mathrm{Spec}\,B' \to \mathcal{X}'$ satisfies the previous condition. Since $\alpha(\xi) \otimes_B B' \simeq \alpha(\xi')$ we get the desired result. □

Remark 1.16. There are set-theoretic problems in considering global sections of presheaves and therefore push-forwards, because $\mathrm{Mod}\,\mathcal{O}_{\mathcal{X}}$ is in general not locally small. The common way to solve this problem is to use Grothendieck universes. Take a universe U and define rings inside U, so that Aff/R is small (with respect to a bigger universe). Fibered categories should then be required to be small too. In this situation it is easy to show that $\mathrm{Mod}\,\mathcal{O}_{\mathcal{X}}$ is locally small and therefore global sections and push-forwards are well defined. With this approach we have to be careful in considering (big) rings defined starting from some $\mathcal{F} \in \mathrm{Mod}\,\mathcal{O}_{\mathcal{X}}$: for instance $\mathrm{Spec}\,\mathcal{O}_{\mathcal{X}}(\mathcal{X})$ is in general not an object of Aff/R.

Notice that global sections and pushforwards of quasi-coherent sheaves are always well defined for a pseudo-algebraic fibered category and a pseudo-algebraic map respectively. The reason is that if $\mathcal{F} \in \operatorname{QCoh} \mathcal{X}$ and $p\colon X \to \mathcal{X}$ is a fpqc atlas then $\mathcal{F}(\mathcal{X}) \to (p^*\mathcal{F})(X)$ is injective and thus $\mathcal{F}(\mathcal{X})$ is a set.

In the rest of the paper we will not be concerned about those set-theoretic problems.

Definition 1.17. If A is an R-algebra and $\mathcal{F} \in \operatorname{Mod} \mathcal{O}_\mathcal{X}$ then an A-module structure on $\mathcal{F}$ is an R-algebra homomorphism $A \to \operatorname{End}_\mathcal{X}(\mathcal{F})$. This is the same data of A-module structures on $\mathcal{F}(\xi)$ commuting with the $\mathrm{H}^0(\mathcal{O}_{\pi(\xi)})$-module structure on $\mathcal{F}(\xi)$ for all $\xi \in \mathcal{X}$ and such that, for all $\xi \to \eta$ in $\mathcal{X}$, the map $\mathcal{F}(\eta) \to \mathcal{F}(\xi)$ is A-linear. We define $\operatorname{QCoh}_A \mathcal{X}$ as the category of quasi-coherent sheaves over $\mathcal{X}$ with an A-module structure. We also define $\mathcal{X}_A$ as the fiber product $\operatorname{Spec} A \times_R \mathcal{X}$.

Notice that if $\mathcal{Y} \to \mathcal{X}$ is a map of fibered categories and $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules with an A-module structure then $g^*\mathcal{F}$ inherits an A-module structure. In particular $g^*\colon \operatorname{QCoh} \mathcal{X} \to \operatorname{QCoh} \mathcal{Y}$ extends to a functor $\operatorname{QCoh}_A \mathcal{X} \to \operatorname{QCoh}_A \mathcal{Y}$.

Proposition 1.18. *Let A be an R-algebra. Then the push-forward map $\operatorname{QCoh} \mathcal{X}_A \to \operatorname{QCoh} \mathcal{X}$ extends naturally to an equivalence $\operatorname{QCoh} \mathcal{X}_A \to \operatorname{QCoh}_A \mathcal{X}$.*

Proof. The result is very simple if $\mathcal{X}$ is an affine scheme. In general, if we set $g\colon \mathcal{X}_A \to \mathcal{X}$ for the projection and we consider $\mathcal{G} \in \operatorname{QCoh} \mathcal{X}_A$, then $g_*\mathcal{G} \in \operatorname{QCoh} \mathcal{X}$ and it inherits an A-module structure from the action of A on $\mathcal{G}$. Therefore $g_*\mathcal{G} \in \operatorname{QCoh}_A \mathcal{X}$. If $h\colon \operatorname{Spec} B \to \mathcal{X}$ is a map consider the diagrams

$$\begin{array}{ccc} \operatorname{Spec}(B \otimes_R A) & \xrightarrow{h'} & \mathcal{X}_A \\ \downarrow g' & & \downarrow g \\ \operatorname{Spec} B & \xrightarrow{h} & \mathcal{X} \end{array} \qquad \begin{array}{ccc} \operatorname{QCoh} \mathcal{X}_A & \xrightarrow{h'^*} & \operatorname{QCoh} \operatorname{Spec}(B \otimes_R A) \\ \downarrow g_* & & \downarrow g'_* \\ \operatorname{QCoh}_A \mathcal{X} & \xrightarrow{h^*} & \operatorname{QCoh}_A \operatorname{Spec} B \end{array}$$

The second diagram is 2-commutative and the last vertical map is an equivalence. Using those diagrams it is easy to define a quasi-inverse $\operatorname{QCoh}_A \mathcal{X} \to \operatorname{QCoh} \mathcal{X}_A$ of g_*. □

We will almost always regard quasi-coherent sheaves over $\mathcal{X}_A$ as objects of $\operatorname{QCoh}_A \mathcal{X}$.

Remark 1.19. The category $\operatorname{Mod} \mathcal{O}_\mathcal{X}$ is symmetric monoidal: if $\mathcal{F}, \mathcal{G} \in \operatorname{Mod} \mathcal{O}_\mathcal{X}$ then the formula

$$(\mathcal{F} \otimes \mathcal{G})(\xi) = \mathcal{F}(\xi) \otimes \mathcal{G}(\xi)$$

defines a presheaf of $\mathcal{O}_\mathcal{X}$-module. Tensor products of quasi-coherent sheaves are quasi-coherent.

If $\mathcal{F}, \mathcal{G} \in \operatorname{QCoh}_A \mathcal{X}$ then $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ does not correspond to the tensor product in $\operatorname{QCoh} \mathcal{X}_A$, but to the tensor product of their pushforward along $\mathcal{X}_A \to \mathcal{X}$. The sheaf $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ has two distinct A-module structures. Under the equivalence $\operatorname{QCoh} \mathcal{X}_A \to \operatorname{QCoh}_A \mathcal{X}$ the tensor product of $\mathcal{F}$ and $\mathcal{G}$, that we will denote by $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{G}$, is given by

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathrm{H}^0(\mathcal{O}_U)} \mathcal{G}(U)/\langle ax \otimes y - x \otimes ay \mid x \in \mathcal{F}(U), y \in \mathcal{G}(U)\rangle$$

Definition 1.20. A locally free sheaf or vector bundle $\mathcal{E}$ (of rank n) over $\mathcal{X}$ is a quasi-coherent sheaf such that $\mathcal{E}(\operatorname{Spec} B \to \mathcal{X})$ is a finitely generated projective B-module (of rank n) for all maps $\operatorname{Spec} B \to \mathcal{X}$. We denote by $\operatorname{Vect} \mathcal{X}$ the subcategory of $\operatorname{QCoh} \mathcal{X}$ of locally free sheaves.

2. Sheafification functors.

In this section we define and describe particular functors that generalize sheafification functors for affine schemes or projective schemes. The idea is to interpret the category of modules or graded modules respectively as a category of R-linear functors. More precisely:

Definition 2.1. Given a fibered category $\mathcal{X}$ over a ring R, an R-algebra A and a subcategory $\mathcal{D}$ of $\operatorname{QCoh}\mathcal{X}$ we define $\mathrm{L}_R(\mathcal{D}, A)$ as the category of contravariant R-linear functors $\Gamma\colon \mathcal{D} \to \operatorname{Mod} A$ and natural transformations as arrows. We define a functor $\Omega^*\colon \operatorname{QCoh}_A \mathcal{X} \to \mathrm{L}_R(\mathcal{D}, A)$ by

$$\Omega^{\mathcal{F}}_{-} = \operatorname{Hom}_{\mathcal{X}}(-, \mathcal{F})\colon \mathcal{D} \to \operatorname{Mod} A$$

The functor Ω^* is called the *Yoneda functor* associated with $\mathcal{D}$. A left adjoint of Ω^* is called a *sheafification* functor associated with $\mathcal{D}$. If $\mathcal{F} \in \operatorname{QCoh}_A \mathcal{X}$ we will call $\Omega^{\mathcal{F}}$ the Yoneda functor associated with $\mathcal{F}$.

The analogy with the sheafification functor is described in Section 7.

Let us fix an R-algebra A and a fibered category $\pi\colon \mathcal{X} \to \operatorname{Aff}/R$.

2.1. **Sheafifying R-linear functors.** In this section we want to explicitly describe sheafification functors for small subcategories of $\operatorname{QCoh}\mathcal{X}$. In particular we fix a *small* (and non empty) subcategory $\mathcal{C}$ of $\operatorname{QCoh}\mathcal{X}$.

In the construction of the sheafification functors we will make use of the coend construction in the settings of categories enriched over categories of modules over a ring. The general theory simplifies considerably in this context and we will also apply such construction only in particular cases. In the following remark we collect all the properties we will need.

Remark 2.2. Let $\mathcal{Y}$ be a fibered category over R, $F\colon \mathcal{C} \to \operatorname{QCoh}\mathcal{Y}$ be an R-linear functor and $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$. The coend of the R-linear functor $\Gamma_- \otimes_R F_-\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \operatorname{QCoh}_A \mathcal{Y}$, denoted by

$$\int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \in \operatorname{QCoh}_A \mathcal{Y}$$

is the cokernel of the map

$$\bigoplus_{\mathcal{E}\to\overline{\mathcal{E}}} (\Gamma_u \otimes \mathrm{id}_{F_{\mathcal{E}}} - \mathrm{id}_{\Gamma_{\overline{\mathcal{E}}}} \otimes F_u)\colon \bigoplus_{\mathcal{E}\to\overline{\mathcal{E}}} \Gamma_{\overline{\mathcal{E}}} \otimes_R F_{\mathcal{E}} \to \bigoplus_{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$$

Moreover it comes equipped with an A-linear natural isomorphism

$$\operatorname{Hom}_{\operatorname{QCoh}_A \mathcal{Y}}(\int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}, \mathcal{H}) \to \operatorname{Hom}_{\mathrm{L}_R(\mathcal{C},A)}(\Gamma, \operatorname{Hom}_{\mathcal{Y}}(F_-, \mathcal{H})) \text{ for } \mathcal{H} \in \operatorname{QCoh}_A \mathcal{Y}$$

The proof of this is just observing that data of a map $\omega\colon \Gamma_{\mathcal{E}} \otimes_R \mathcal{F}_{\mathcal{E}} \to \mathcal{H}$ and a map $\tilde{\omega}\colon \Gamma_{\mathcal{E}} \to \operatorname{Hom}_{\mathcal{Y}}(\mathcal{F}_{\mathcal{E}}, \mathcal{H})$ are the same, and the condition that $\tilde{\omega}$ is a natural transformation is exactly the condition that ω pass to the quotient $\int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$. Everything can be written down by the following expression:

$$\alpha(\int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \to \mathcal{H})(x) = \omega \circ p_{\overline{\mathcal{E}}}(x \otimes -)\colon F_{\overline{\mathcal{E}}} \to \mathcal{H} \text{ for } \overline{\mathcal{E}} \in \mathcal{C}, x \in \Gamma_{\overline{\mathcal{E}}}$$

where $p_{\mathcal{E}}\colon \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \to \int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$ for $\mathcal{E} \in \mathcal{C}$ are the structure morphisms. Its inverse is uniquely determined by the expression

$$\alpha^{-1}(\Gamma \to \operatorname{Hom}_{\mathcal{Y}}(F_-, \mathcal{H})) \circ p_{\mathcal{E}}\colon \Gamma_{\mathcal{E}} \otimes_R \mathcal{E} \to \mathcal{H},\ x \otimes y \longmapsto v_{\mathcal{E}}(x)(y) \text{ for } \mathcal{E} \in \mathcal{C}$$

Natural transformations $F \to F'$ and $\Gamma \to \Gamma'$ yields morphisms $\int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \to \int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F'_{\mathcal{E}}$ and $\int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}} \to \int^{\mathcal{E}\in\mathcal{C}} \Gamma'_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$ respectively. Those can be defined either using Yoneda's lemma and the above characterization of $\operatorname{Hom}(\int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}, -)$ or directly using the description of $\int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R F_{\mathcal{E}}$ as a cokernel.

All the above claims are standard in the theory of coend in the enriched settings (in our case enriched by $\operatorname{Mod} R$), but, in this simplified context, it is elementary to prove them directly.

We start by showing that $\mathcal{C}$ (and therefore any essentially small subcategory of $\mathrm{QCoh}\,\mathcal{X}$) admits a sheafification functor.

Proposition 2.3. *The Yoneda functor* $\Omega^*\colon \mathrm{QCoh}_A\,\mathcal{X} \to \mathrm{L}_R(\mathcal{C}, A)$ *has a left adjoint* $\mathcal{F}_{-,\mathcal{C}}\colon \mathrm{L}_R(\mathcal{C}, A) \to \mathrm{QCoh}_A\,\mathcal{X}$ *given by*

$$\mathcal{F}_{\Gamma,\mathcal{C}} = \int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R \mathcal{E} \in \mathrm{QCoh}_A\,\mathcal{X}$$

where $\mathcal{E}$ *denotes the inclusion* $\mathcal{C} \to \mathrm{QCoh}\,\mathcal{X}$*. Given* $\xi \in \mathcal{X}$ *we have that*

$$\mathcal{F}_{\Gamma,\mathcal{C}}(\xi) = \int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R \mathcal{E}(\xi) \in \mathrm{Mod}(\mathrm{H}^0(\mathcal{O}_{\pi(\xi)}) \otimes_R A)$$

where $\mathcal{E}(\xi)$ *denotes the evaluation* $\mathcal{C} \to \mathrm{Mod}\,\mathrm{H}^0(\mathcal{O}_{\pi(\xi)})$ *of sheaves in* $\xi \in \mathcal{X}$.

Proof. It is enough to apply 2.2 with $\mathcal{Y} = \mathcal{X}$ and $F\colon \mathcal{C} \to \mathrm{QCoh}\,\mathcal{X}$ the inclusion. Using the description of coend as cokernel one can check that the two functors defined in the statement are canonically isomorphic. □

Remark 2.4. Given $\mathcal{H} \in \mathrm{QCoh}_A\,\mathcal{X}$ and $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ the adjunction is

$$\mathrm{Hom}_{\mathcal{X}_A}(\mathcal{F}_{\Gamma,\mathcal{C}}, \mathcal{H}) \simeq \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},A)}(\Gamma, \mathrm{Hom}_{\mathcal{X}}(-, \mathcal{H}))$$

Explicitly this map is just

$$(\Gamma_{\mathcal{E}} \otimes \mathcal{E} \to \mathcal{H}) \longmapsto (\Gamma_{\mathcal{E}} \to \mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{H})) \text{ for } \mathcal{E} \in \mathcal{C}$$

where we think of $\mathcal{F}_{\Gamma,\mathcal{C}}$ as a quotient of $\bigoplus_{\mathcal{E}} \Gamma_{\mathcal{E}} \otimes \mathcal{E}$ as in 2.2.

Definition 2.5. We denote by $\gamma_\Gamma\colon \Gamma_- \to \Omega_-^{\mathcal{F}_{\Gamma,\mathcal{C}}} = \mathrm{Hom}_{\mathcal{X}}(-, \mathcal{F}_{\Gamma,\mathcal{C}})$ and $\delta_{\mathcal{G}}\colon \mathcal{F}_{\Omega^{\mathcal{G}},\mathcal{C}} \to \mathcal{G}$ for $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ and $\mathcal{G} \in \mathrm{QCoh}_A\,\mathcal{X}$ the unit and the counit of the adjunction between $\Omega^*\colon \mathrm{QCoh}_A\,\mathcal{X} \to \mathrm{L}_R(\mathcal{C}, A)$ and $\mathcal{F}_{*,\mathcal{C}}\colon \mathrm{L}_R(\mathcal{C}, A) \to \mathrm{QCoh}_A\,\mathcal{X}$ respectively.

Given $\xi \in \mathcal{X}$, $\mathcal{E} \in \mathcal{C}$, $\psi \in \mathcal{E}(\xi)$ and $x \in \Gamma_{\mathcal{E}}$ we denote by $x_{\mathcal{E},\psi} \in \mathcal{F}_{\Gamma,\mathcal{C}}(\xi)$ the image of $x \otimes \psi$ under the map $\Gamma_{\mathcal{E}} \otimes_R \mathcal{E}(\xi) \to \mathcal{F}_{\Gamma,\mathcal{C}}(\xi)$

Proposition 2.6. *Let* $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$*. The unit* $\gamma_\Gamma\colon \Gamma_- \to \Omega_-^{\mathcal{F}_{\Gamma,\mathcal{C}}} = \mathrm{Hom}_{\mathcal{X}}(-, \mathcal{F}_{\Gamma,\mathcal{C}})$ *is given by*

$$\begin{aligned} \Gamma_{\mathcal{E}} &\longrightarrow \mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}_{\Gamma,\mathcal{C}}) \\ x &\longmapsto (\phi \longmapsto x_{\mathcal{E},\phi}) \end{aligned}$$

If $\mathcal{G} \in \mathrm{QCoh}_A\,\mathcal{X}$ *the counit* $\delta_{\mathcal{G}}\colon \mathcal{F}_{\Omega^{\mathcal{G}},\mathcal{C}} \to \mathcal{G}$ *is given by*

$$\mathcal{F}_{\Omega^{\mathcal{G}},\mathcal{C}}(\xi) \ni x_{\mathcal{E},\psi} \longmapsto x(\psi) \in \mathcal{G}(\xi) \text{ for } \mathcal{E} \in \mathcal{C}, x \in \Omega_{\mathcal{E}}^{\mathcal{G}} = \mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{G}), \xi \in \mathcal{X}, \psi \in \mathcal{E}(\xi)$$

Proof. This follows easily from the description in 2.4. □

Given a map $g\colon \mathcal{Y} \to \mathcal{X}$ of fibered categories we want to express $g^*\mathcal{F}_{\Gamma,\mathcal{C}} \in \mathrm{QCoh}_A\,\mathcal{Y}$ for $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ as $\mathcal{F}_{g^*\Gamma,g^*\mathcal{C}}$ for a suitable choice of $g^*\mathcal{C} \subseteq \mathrm{QCoh}\,\mathcal{Y}$ and $g^*\Gamma \in \mathrm{L}_R(g^*\mathcal{C}, A)$.

Definition 2.7. Let $\mathcal{Y}$ be a fibered category, $g\colon \mathcal{Y} \to \mathcal{X}$ be a morphism and $\mathcal{D}$ be a subcategory of $\mathrm{QCoh}\,\mathcal{X}$. We set $g^*\mathcal{D}$ for the subcategory of $\mathrm{QCoh}\,\mathcal{Y}$ of sheaves $g^*\mathcal{E}$ for $\mathcal{E} \in \mathcal{D}$. If $\mathcal{D}' \subseteq \mathrm{QCoh}\,\mathcal{Y}$ is a subcategory containing $g^*\mathcal{D}$ we can define a restriction functor

$$\begin{aligned} \mathrm{L}_R(\mathcal{D}', A) &\xrightarrow{g_*} \mathrm{L}_R(\mathcal{D}, A) \\ \Gamma &\longmapsto \Gamma \circ g^* \end{aligned}$$

Proposition 2.8. *Let $\mathcal{Y}$ be a fibered category, $g\colon \mathcal{Y} \to \mathcal{X}$ be a morphism and $\mathcal{D}$ be a subcategory of $\mathrm{QCoh}\,\mathcal{Y}$ such that $g^*\mathcal{C} \subseteq \mathcal{D}$. Then $g_*\colon \mathrm{L}_R(\mathcal{D},A) \to \mathrm{L}_R(\mathcal{C},A)$ has a left adjoint $g^*\colon \mathrm{L}_R(\mathcal{C},A) \to \mathrm{L}_R(\mathcal{D},A)$ and it is given by*

$$(g^*\Gamma)_{\mathcal{G}} = \int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes_R \mathrm{Hom}_{\mathcal{Y}}(\mathcal{G}, g^*\mathcal{E}) \in \mathrm{Mod}\,A \text{ for } \Gamma \in \mathrm{L}_R(\mathcal{C},A),\ \mathcal{G} \in \mathcal{D}$$

where $\mathrm{Hom}_{\mathcal{Y}}(\mathcal{G}, g^*-)$ is thought of as a functor $\mathcal{C} \to \mathrm{Mod}\,R$. If $\mathcal{Y} = \mathcal{X}$ and $g = \mathrm{id}_{\mathcal{X}}$, so that $\mathcal{C} \subseteq \mathcal{D}$ and $(\mathrm{id}_{\mathcal{X}})_*\colon \mathrm{L}_R(\mathcal{D},A) \to \mathrm{L}_R(\mathcal{C},A)$ is the restriction, then the unit $\Gamma \to (\mathrm{id}_{\mathcal{X}}^*\Gamma)_{|\mathcal{C}}$ is an isomorphism for $\Gamma \in \mathrm{L}_R(\mathcal{C},A)$.

Proof. Let $\Omega \in \mathrm{L}_R(\mathcal{D},A)$, $\Gamma \in \mathrm{L}_R(\mathcal{C},A)$ and $\mathcal{G} \in \mathcal{D}$. Applying 2.2 with $F = \mathrm{Hom}_{\mathcal{Y}}(\mathcal{G}, g^*-)\colon \mathcal{C} \to \mathrm{Mod}(R)$ and $\Gamma\colon \mathcal{C} \to \mathrm{Mod}(A)$ we get a bijection between A-linear maps $(g^*\Gamma)_{\mathcal{G}} \to \Omega_{\mathcal{G}}$ and the set of A-linear natural transformations $\Gamma_{\mathcal{E}} \to \mathrm{Hom}_R(\mathrm{Hom}(\mathcal{G}, g^*\mathcal{E}), \Omega_{\mathcal{G}})$ for $\mathcal{E} \in \mathcal{C}$. Thinking of $(g^*\Gamma)_{\mathcal{G}}$ as a quotient of $\bigoplus_{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes \mathrm{Hom}(\mathcal{G}, g^*\mathcal{E})$ the previous map is just the canonical map

$$\Gamma_{\mathcal{E}} \to \mathrm{Hom}_R(\mathrm{Hom}(\mathcal{G}, g^*\mathcal{E}), \Omega_{\mathcal{G}}) \longmapsto \Gamma_{\mathcal{E}} \otimes \mathrm{Hom}(\mathcal{G}, g^*\mathcal{E}) \to \Omega_{\mathcal{G}}$$

With this description in mind it is elementary to check that a natural transformation $g^*\Gamma \to \Omega$ corresponds to a collection $\underline{\gamma}$ of A-linear maps $\gamma_{\mathcal{G},\mathcal{E}}\colon \Gamma_{\mathcal{E}} \to \mathrm{Hom}_R(\mathrm{Hom}(\mathcal{G}, g^*\mathcal{E}), \Omega_{\mathcal{G}})$ such that

$$\gamma_{\mathcal{G},\mathcal{E}}(x)(\phi \circ u) = \Omega_u(\gamma_{\overline{\mathcal{G}},\mathcal{E}}(x)(\phi)) \text{ for } x \in \Gamma_{\mathcal{E}}, \phi\colon \overline{\mathcal{G}} \to g^*\mathcal{E}, u\colon \mathcal{G} \to \overline{\mathcal{G}} \in \mathcal{D}$$

$$\gamma_{\mathcal{G},\mathcal{E}'}(\Gamma_v(x))(\phi') = \gamma_{\mathcal{G},\mathcal{E}}(x)(g^*v \circ \phi') \text{ for } x \in \Gamma_{\mathcal{E}}, \phi'\colon \mathcal{G} \to g^*\mathcal{E}', v\colon \mathcal{E}' \to \mathcal{E}$$

The first is the functoriality in $\mathcal{G}$, the second in $\mathcal{E}$. Given any collection of maps $\gamma_{\mathcal{G},\mathcal{E}}\colon \Gamma_{\mathcal{E}} \to \mathrm{Hom}_R(\mathrm{Hom}(\mathcal{G}, g^*\mathcal{E}), \Omega_{\mathcal{G}})$ define

$$\mu(\gamma)_{\mathcal{E}}\colon \Gamma_{\mathcal{E}} \to \Omega_{g^*\mathcal{E}},\ \mu(\gamma)_{\mathcal{E}}(x) = \gamma_{g^*\mathcal{E},\mathcal{E}}(x)(\mathrm{id}_{g^*\mathcal{E}})$$

Conversely, given any collection of maps $\mu_{\mathcal{E}}\colon \Gamma_{\mathcal{E}} \to \Omega_{g^*\mathcal{E}}$ define

$$\gamma(\mu)_{\mathcal{G},\mathcal{E}}\colon \Gamma_{\mathcal{E}} \to \mathrm{Hom}_R(\mathrm{Hom}(\mathcal{G}, g^*\mathcal{E}), \Omega_{\mathcal{G}}),\ \gamma(\mu)_{\mathcal{G},\mathcal{E}}(x)(\psi) = \Omega_\psi(\mu(x))$$

Now come the tedious verification that $\gamma_{\mathcal{G},\mathcal{E}}$ yields a natural transformation $g^*\Gamma \to \Omega$ if and only if $\mu_{\mathcal{E}}$ define a natural transformation $\Gamma \to g_*\Omega$ and that this defines a bijection $\mathrm{Hom}(g^*\Gamma, \Omega) \simeq \mathrm{Hom}(\Gamma, g_*\Omega)$. This is elementary and can be carried on just using symbols and expressions above.

Assume now $\mathcal{Y} = \mathcal{X}$ and $g = \mathrm{id}_{\mathcal{X}}$ and let $\Gamma \in \mathrm{L}_R(\mathcal{C},A)$ and $\overline{\mathcal{E}} \in \mathcal{C}$. Denote by $\alpha\colon \Gamma \to (\mathrm{id}_{\mathcal{X}}^*\Gamma)_{|\mathcal{C}}$ the unit morphism. If $p_{\widetilde{\mathcal{E}}}\colon \Gamma_{\widetilde{\mathcal{E}}} \otimes \mathrm{Hom}_{\mathcal{X}}(\overline{\mathcal{E}}, \widetilde{\mathcal{E}}) \to (\mathrm{id}_{\mathcal{X}}^*\Gamma)_{\overline{\mathcal{E}}}$ are the structure morphisms as in 2.2, then

$$\alpha_{\overline{\mathcal{E}}}\colon \Gamma_{\overline{\mathcal{E}}} \to (\mathrm{id}_{\mathcal{X}}^*\Gamma)_{\overline{\mathcal{E}}},\ \alpha_{\overline{\mathcal{E}}}(x) = p_{\overline{\mathcal{E}}}(x \otimes \mathrm{id}_{\overline{\mathcal{E}}})$$

In particular, given $\mathcal{H} \in \mathrm{Mod}\,A$ and using 2.2, the map $\mathrm{Hom}_A(\alpha_{\overline{\mathcal{E}}}, \mathcal{H})\colon \mathrm{Hom}_A((\mathrm{id}_{\mathcal{X}}^*\Gamma)_{\overline{\mathcal{E}}}, \mathcal{H}) \to \mathrm{Hom}_A(\Gamma_{\overline{\mathcal{E}}}, \mathcal{H})$ sends an A-linear natural transformation $\delta\colon \Gamma_- \to \mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{X}}(\overline{\mathcal{E}}, -), \mathcal{H})$ to $\Gamma_{\overline{\mathcal{E}}} \ni x \longmapsto \delta(x)(\mathrm{id}_{\overline{\mathcal{E}}}) \in \mathcal{H}$. Since δ corresponds to an R-linear natural transformation $\mathrm{Hom}_{\mathcal{X}}(\overline{\mathcal{E}}, -) \to \mathrm{Hom}_A(\Gamma_-, \mathcal{H})$, we can rewrite $\mathrm{Hom}_A(\alpha_{\overline{\mathcal{E}}}, \mathcal{H})$ as

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\mathrm{Hom}(\overline{\mathcal{E}}, -), \Delta) \to \Delta_{\overline{\mathcal{E}}},\ \omega \mapsto \omega(\mathrm{id}_{\overline{\mathcal{E}}}),\ \Delta = \mathrm{Hom}_A(\Gamma_*, \mathcal{H})$$

By enriched Yoneda lemma we can conclude that $\mathrm{Hom}_A(\alpha_{\overline{\mathcal{E}}}, \mathcal{H})$ is an isomorphism and, therefore, that $\alpha_{\overline{\mathcal{E}}}$ is an isomorphism as required. □

Remark 2.9. Proposition above can also be reinterpreted using Kan extension. Indeed given an R-linear functor $\Gamma\colon \mathcal{C} \to (\mathrm{Mod}\,A)^{\mathrm{op}}$ (notice the opposite category) and $g^*\colon \mathcal{C} \to \mathcal{D}$ then $g^*\Gamma\colon \mathcal{C} \to (\mathrm{Mod}\,A)^{\mathrm{op}}$ is the right Kan extension of Γ along g^*. This follows from the definition of the Kan extension and the adjointness we proved. In particular one could have proved proposition above using standard results about cocomplete categories. On the other hand we need the coend description.

The above proposition yields a natural extension of any $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ to a functor $\Gamma^{ex} \in \mathrm{L}_R(\mathrm{QCoh}\,\mathcal{X}, A)$. By abuse of notation we will denote them by the same symbol Γ. This means that if $\Gamma \in L_R(\mathcal{C}, A)$ and $\mathcal{G} \in \mathrm{QCoh}\,\mathcal{X}$ then we can evaluate Γ on $\mathcal{G}$, writing $\Gamma_{\mathcal{G}}$.

Given a map $g\colon \mathcal{Y} \to \mathcal{X}$ we will denote by $g^*\colon \mathrm{L}_R(\mathcal{C}, A) \to \mathrm{L}_R(g^*\mathcal{C}, A)$ the left adjoint of the restriction $\mathrm{L}_R(g^*\mathcal{C}, A) \to \mathrm{L}_R(\mathcal{C}, A)$. So, given $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$, $g^*\Gamma$ is a functor $g^*\mathcal{C} \to \mathrm{Mod}\,A$ but it also defines a functor $\mathrm{QCoh}\,\mathcal{Y} \to \mathrm{Mod}\,A$ denoted, by our convention, by the same symbol. By 2.8 the functor $g^*\Gamma\colon \mathrm{QCoh}\,\mathcal{Y} \to \mathrm{Mod}\,A$ coincides with the value of the left adjoint of the restriction $\mathrm{L}_R(\mathrm{QCoh}\,\mathcal{Y}, A) \to \mathrm{L}_R(\mathcal{C}, A)$.

Remark 2.10. Given $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ and $\mathcal{E} \in \mathcal{C}$ we have R-linear morphisms of rings

$$\mathrm{H}^0(\mathcal{O}_{\mathcal{X}}) \simeq \mathrm{End}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{End}_{\mathcal{X}}(\mathcal{E}) \to \mathrm{End}_A(\Gamma_{\mathcal{E}})$$

This defines a lifting of Γ to an R-linear functor $\Gamma\colon \mathcal{C} \to \mathrm{Mod}(\mathrm{H}^0(\mathcal{O}_{\mathcal{X}}) \otimes_R A)$ and an equivalence

$$\mathrm{L}_R(\mathcal{C}, A) \to \mathrm{L}_R(\mathcal{C}, \mathrm{H}^0(\mathcal{O}_{\mathcal{X}}) \otimes_R A)$$

In particular, if $g\colon \mathrm{Spec}\,B \to \mathcal{X}$ is a map and $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ then $(g^*\Gamma)_B$ has a $B \otimes_R A$-module structure. By 2.3 and 2.8 there is a canonical A-linear isomorphism

$$\mathcal{F}_{\Gamma,\mathcal{C}}(B) \simeq (g^*\Gamma)_B$$

and it is easy to see that it is also B-linear.

Proposition 2.11. *Let $g\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of fibered categories and $\mathcal{D} \subseteq \mathrm{QCoh}(\mathcal{Y})$ such that $g^*\mathcal{C} \subseteq \mathcal{D}$. If $g^*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ has a right adjoint $g_*\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ then there is a canonical isomorphism $g_*(\Omega^{\mathcal{N}}_{|\mathcal{D}}) \simeq \Omega^{g_*\mathcal{N}}_{|\mathcal{C}}$ for $\mathcal{N} \in \mathrm{QCoh}_A\,\mathcal{Y}$,that is a 2-commutative diagram*

$$\begin{array}{ccc} \mathrm{QCoh}_A(\mathcal{Y}) & \xrightarrow{\Omega^*} & \mathrm{L}_R(\mathcal{D}, A) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ \mathrm{QCoh}_A(\mathcal{X}) & \xrightarrow{\Omega^*} & \mathrm{L}_R(\mathcal{C}, A) \end{array}$$

Moreover, without any assumption of g, but assuming $\mathcal{D}$ small, there exists an isomorphism $g^*\mathcal{F}_{\Gamma,\mathcal{C}} \simeq \mathcal{F}_{g^*\Gamma, g^*\mathcal{C}}$ natural in $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$, that is a 2-commutative diagram

$$\begin{array}{ccc} \mathrm{L}_R(\mathcal{C}, A) & \xrightarrow{\mathcal{F}_{-,\mathcal{C}}} & \mathrm{QCoh}_A\,\mathcal{X} \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\ \mathrm{L}_R(\mathcal{D}, A) & \xrightarrow{\mathcal{F}_{-,\mathcal{D}}} & \mathrm{QCoh}_A\,\mathcal{Y} \end{array}$$

Concretely the isomorphism is of the form

$$\begin{array}{ccc} g^*(\Gamma_{\mathcal{E}} \otimes \mathcal{E}) & \longrightarrow & (g*\Gamma)_{g^*\mathcal{E}} \otimes g^*\mathcal{E} \\ \downarrow & & \downarrow \\ g^*\mathcal{F}_{\Gamma,\mathcal{C}} & \longrightarrow & \mathcal{F}_{g^*\Gamma,\mathcal{D}} \end{array}$$

where $\Gamma_{\mathcal{E}} \to (g^*\Gamma)_{g^*\mathcal{E}} = (g_*g^*\Gamma)_{\mathcal{E}}$ is the adjunction map. In particular the isomorphism $g^*\mathcal{F}_{\Gamma,\mathcal{C}} \to \mathcal{F}_{g^*\Gamma, g^*\mathcal{C}}$ is compatible with composition of maps $g\colon \mathcal{Y} \to \mathcal{X}$.

Proof. The first isomorphism is clear because

$$\Omega^{g_*\mathcal{N}}_{\mathcal{E}} = \mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, g_*\mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{Y}}(g^*\mathcal{E}, \mathcal{N}) = \Omega^{\mathcal{N}}_{g^*\mathcal{E}} = (g_*\Omega^{\mathcal{N}})_{\mathcal{E}}$$

In particular, assuming the existence of g_*, the second part of the statement is a consequence of the first by adjointness. We have to prove the second part without this.

Let $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$, $\xi\colon \operatorname{Spec} B \to \mathcal{Y}$ be a map and $N \in \operatorname{Mod} B \otimes_R A$. Denote by $F\colon \mathcal{C} \to \operatorname{Mod} B$ and $G\colon \mathcal{D} \to \operatorname{Mod} B$ the functors obtained evaluating the sheaves in $g\xi$ and ξ respectively. In particular $F = G \circ g^* = g_* G$. We have

$$(g^*\mathcal{F}_{\Gamma,\mathcal{C}})(B) = \mathcal{F}_{\Gamma,\mathcal{C}}(g\xi) = \int^{\mathcal{E}\in\mathcal{C}} \Gamma_{\mathcal{E}} \otimes F_{\mathcal{E}} \text{ and } \mathcal{F}_{g^*\Gamma,\mathcal{D}}(B) = \int^{\mathcal{H}\in\mathcal{D}} (g^*\Gamma)_{\mathcal{H}} \otimes G_{\mathcal{H}}$$

They implies respectively that

$$\operatorname{Hom}_{B\otimes_R A}((g^*\mathcal{F}_{\Gamma,\mathcal{C}})(B), N) \simeq \operatorname{Hom}_{\mathrm{L}_R(\mathcal{C},A)}(\Gamma, \operatorname{Hom}_B(F, N))$$
$$\operatorname{Hom}_{B\otimes_R A}(\mathcal{F}_{g^*\Gamma,\mathcal{D}}(B), N) \simeq \operatorname{Hom}_{\mathrm{L}_R(\mathcal{D},A)}(g^*\Gamma, \operatorname{Hom}_B(G, N))$$

Since $\operatorname{Hom}_B(F, N) = g_* \operatorname{Hom}_B(G, N)$ the above modules are canonically isomorphic. In particular we get an isomorphism $(g^*\mathcal{F}_{\Gamma,\mathcal{C}})(B) \simeq \mathcal{F}_{g^*\Gamma,\mathcal{D}}(B)$. By a direct check we see that this map fits in the last commutative diagram in the statement evaluated in $\xi\colon \operatorname{Spec} B \to \mathcal{Y}$. This shows that $g^*\mathcal{F}_{\Gamma,\mathcal{C}} \to \mathcal{F}_{g^*\Gamma,\mathcal{D}}$ is well defined and an isomorphism. Naturality in $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ also follows easily. ☐

Remark 2.12. For $g = \mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ and $\mathcal{C} \subseteq \mathcal{D}$ proposition above is telling us that if $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ and we extend it to $\Gamma^{ex} \in \mathrm{L}_R(\mathcal{D}, A)$ then $\mathcal{F}_{\Gamma,\mathcal{C}} \simeq \mathcal{F}_{\Gamma^{ex},\mathcal{D}}$.

Remark 2.13. If $A \to A'$ is a morphism of R-algebras then we have pull-back functors $\mathrm{L}_R(\mathcal{C}, A) \to \mathrm{L}_R(\mathcal{C}, A')$ and $\operatorname{QCoh}_A \mathcal{X} \to \operatorname{QCoh}_{A'} \mathcal{X}$. The first one is obtained considering the tensor product $- \otimes_A A'$, while the second one corresponds to the pullback $\operatorname{QCoh} \mathcal{X}_A \to \operatorname{QCoh} \mathcal{X}_{A'}$ along the projection $\mathcal{X}_{A'} \to \mathcal{X}_A$. Alternatively, those functors are left adjoints to the restriction of scalars $\mathrm{L}_R(\mathcal{C}, A') \to \mathrm{L}_R(\mathcal{C}, A)$ and $\operatorname{QCoh}_{A'} \mathcal{X} \to \operatorname{QCoh}_A \mathcal{X}$ respectively. It is easy to see that in this way we obtain two fpqc stacks (not in groupoids) $\mathrm{L}_R(\mathcal{C}, -)$ and $\operatorname{QCoh}_{\_} \mathcal{X}$ over the category of affine R-schemes. Notice that the functor $\Omega^*\colon \operatorname{QCoh}_{\_} \mathcal{X} \to \mathrm{L}_R(\mathcal{C}, -)$ is not a morphism of stacks because $\operatorname{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{G}) \otimes_A A' \not\simeq \operatorname{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{G} \otimes_A A')$ in general for $\mathcal{E} \in \mathcal{C}$ and $\mathcal{G} \in \operatorname{QCoh}_A \mathcal{X}$.

Proposition 2.14. *The functor* $\mathcal{F}_{*,\mathcal{C}}\colon \mathrm{L}_R(\mathcal{C}, -) \to \operatorname{QCoh}_{\_} \mathcal{X}$ *is a morphism of stacks.*

Proof. Given a morphism $A \to A'$ of R-algebras we have a 2-commutative diagram

$$\begin{array}{ccc} \operatorname{QCoh}_{A'} \mathcal{X} & \xrightarrow{\Omega^*} & \mathrm{L}_R(\mathcal{C}, A') \\ \downarrow & & \downarrow \\ \operatorname{QCoh}_A \mathcal{X} & \xrightarrow{\Omega^*} & \mathrm{L}_R(\mathcal{C}, A) \end{array}$$

where the vertical arrows are obtained by restricting the scalars from A' to A. Using 2.13 and taking the left adjoint functors of the functors in the diagram we exactly get the 2-commutative diagram expressing the fact that $\mathcal{F}_{*,\mathcal{C}}$ preserves Cartesian arrows. ☐

We conclude this section by showing that, when considering sheafification functors $\mathcal{F}_{-,\mathcal{C}}$, we can always reduce problems to the case when $\mathcal{C}$ is an additive category. Moreover in this case the sections of $\mathcal{F}_{-,\mathcal{C}}$ have a nice expression in terms of a direct limit.

Definition 2.15. Given a subcategory $\mathcal{D}$ of $\operatorname{QCoh} \mathcal{X}$ we denote by $\mathcal{D}^{\oplus}$ the subcategory of $\operatorname{QCoh} \mathcal{X}$ whose objects are all finite direct sums of sheaves in $\mathcal{D}$.

Notice that if $\mathcal{D}$ is small then $\mathcal{D}^{\oplus}$ is small.

Proposition 2.16. *Let* $\mathcal{D} \subseteq \operatorname{QCoh}(\mathcal{X})$ *be a subcategory and* $\mathscr{B}$ *be an additive* R*-linear category. Then* $\mathcal{D} \subseteq \mathcal{D}^{\oplus}$ *induces an equivalence between the category of (contravariant)* R*-linear functors* $\mathcal{D}^{\oplus} \to \mathscr{B}$ *and the category of (contravariant)* R*-linear functors* $\mathcal{D} \to \mathscr{B}$*.*

In particular the restriction $\mathrm{L}_R(\mathcal{D}^{\oplus}, A) \to \mathrm{L}_R(\mathcal{D}, A)$ *is an equivalence. If* $\mathcal{D}$ *is small and* $\Gamma \in \mathrm{L}_R(\mathcal{D}^{\oplus}, A)$ *then we have a canonical isomorphism* $\mathcal{F}_{\Gamma,\mathcal{C}^{\oplus}} \simeq \mathcal{F}_{\Gamma_{|\mathcal{C}},\mathcal{C}}$*.*

Proof. The contravariant case follows from the covariant one replacing $\mathscr{B}$ by $\mathscr{B}^{\mathrm{op}}$. Given $\mathcal{E} \in \mathcal{D}^{\oplus}$ let us fix a finite set $I_{\mathcal{E}} \subseteq \mathrm{Obj}(\mathcal{D})$ with an isomorphism $\mathcal{E} \simeq \bigoplus_{V \in I_{\mathcal{E}}} V$. If $\mathcal{E} \in \mathcal{D}$ we choose $I_{\mathcal{E}} = \{\mathcal{E}\}$. Given an R-linear functor $\Gamma \colon \mathcal{D} \to \mathscr{B}$ we define an extension $\Delta^{\Gamma} \colon \mathcal{D}^{\oplus} \to \mathscr{B}$ as follows. On objects we set

$$\Delta^{\Gamma}_{\mathcal{E}} = \bigoplus_{V \in I_{\mathcal{E}}} \Gamma_V \in \mathscr{B}$$

A morphism $\phi \colon \mathcal{E} \to \mathcal{E}'$ in $\mathcal{D}^{\oplus}$ is completely determined by a matrix $(\phi_{V,W})_{V \in I_{\mathcal{E}}, W \in I_{\mathcal{E}'}}$ with entries $\phi_{V,W} \in \mathrm{Hom}_{\mathcal{D}}(V, W)$. We define $\Delta^{\Gamma}_{\mathcal{E}} \to \Delta^{\Gamma}_{\mathcal{E}'}$ as the map corresponding to the matrix $(\Gamma_{\phi_{V,W}})_{V \in I_{\mathcal{E}}, W \in I_{\mathcal{E}'}}$. Using the linearity of Γ we can deduce that $\Delta^{\Gamma} \colon \mathcal{D}^{\oplus} \to \mathscr{B}$ is a well defined R-linear functor extending Γ. Moreover it is easy to see that this is a quasi-inverse to the restriction of functors. Last claim follows from 2.12. □

Remark 2.17. If $\mathscr{C}$ is an R-linear and additive category and $F, G \colon \mathscr{C} \to \mathrm{Mod}\, A$ are R-linear (covariant or contravariant) functors then any natural transformation $\lambda \colon F \to G$ of functors of sets is R-linear. Indeed by considering $\mathscr{C}^{\mathrm{op}}$ we can consider only covariant functors. In this case it is easy to show that the maps $\lambda_X \colon F(X) \to G(X)$ for $X \in \mathscr{C}$ are R-linear using functoriality on the map $r\mathrm{id}_X \colon X \to X$ for $r \in R$ and $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_1 + \mathrm{pr}_2 \colon X \oplus X \to X$, where pr_* are the projections.

Remark 2.18. If I is a small category and $F \colon I \to \mathrm{QCoh}\, \mathcal{Y}$ is a functor (covariant or contravariant), for some category fibered in groupoids $\mathcal{Y}$ over R, then the colimit $\varinjlim_i F(i)$ always exists in $\mathrm{QCoh}(\mathcal{Y})$. Indeed, replacing I by I^{op}, we can assume F contravariant. In that case the colimit is the cokernel of the map

$$\bigoplus_{i \to j} F(j) \to \bigoplus_k F(k), \ (\alpha \colon i \to j, x \in F(j)) \longmapsto F(\alpha)(x) - x \in F(i) \oplus F(j))$$

Moreover for later reference we observe the following. If for all objects $i, j \in I$ there exists $k \in I$ and maps $k \to i, k \to j$ then all elements of $\varinjlim_i F(i)$ comes from some $F(q)$.

Definition 2.19. Let $\mathrm{Spec}\, B \to \mathcal{X}$ be a map. We denote by $J_{B,\mathcal{C}}$ the category of pairs $(\mathcal{E}, \psi)$ where $\mathcal{E} \in \mathcal{C}^{\oplus}$ and $\psi \in \mathcal{E}(B)$. Given $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ we have a functor $\Gamma \colon J_{B,\mathcal{C}} \to \mathrm{Mod}\, A$ given by $\Gamma_{\mathcal{E},\psi} = \Gamma_{\mathcal{E}}$.

We will make colimits over the category $J_{B,\mathcal{C}}$, but we warn the reader that this is not filtered in general.

Proposition 2.20. *Let* $\mathrm{Spec}\, B \to \mathcal{X}$ *be a map and* $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$*. The category* $J_{B,\mathcal{C}}$ *is non-empty and for all* $\xi, \xi' \in J_{B,\mathcal{C}}$ *there exists* $\xi'' \in J_{B,\mathcal{C}}$ *and maps* $\xi'' \to \xi$, $\xi'' \to \xi'$*. The A-linear maps* $\Gamma_{\mathcal{E},\psi} = \Gamma_{\mathcal{E}} \to \mathcal{F}_{\Gamma,\mathcal{C}^{\oplus}}(B) \simeq \mathcal{F}_{\Gamma,\mathcal{C}}(B)$, $x \longmapsto x_{\mathcal{E},\psi}$ *for* $(\mathcal{E}, \psi) \in J_{B,\mathcal{C}}$ *(see 2.5) induce an A-linear isomorphism*

$$\varinjlim_{(\mathcal{E},\psi) \in J^{\mathrm{op}}_{B,\mathcal{C}}} \Gamma_{\mathcal{E},\psi} \to \mathcal{F}_{\Gamma,\mathcal{C}}(B)$$

The multiplication by $b \in B$ *on the first limit is induced by mapping* $\Gamma_{\mathcal{E},\psi}$ *to* $\Gamma_{\mathcal{E},b\psi}$ *using* $\mathrm{id}_{\Gamma_{\mathcal{E}}}$ *for* $(\mathcal{E}, \psi) \in J_{B,\mathcal{C}}$*. In other words*

$$x_{\mathcal{E},b\psi} = b x_{\mathcal{E},\psi} \text{ for } \mathcal{E} \in \mathcal{C}^{\oplus}, x \in \Gamma_{\mathcal{E}}, \psi \in \mathcal{E}(B), b \in B$$

Finally every element of $\mathcal{F}_{\Gamma,\mathcal{C}}(B)$ *is of the form* $x_{\mathcal{E},\psi}$ *for some* $(\mathcal{E}, \psi) \in J_{B,\mathcal{C}}$*.*

Proof. By 2.16 we can assume $\mathcal{C} = \mathcal{C}^{\oplus}$. Denote by $\mathcal{H}$ and $\alpha \colon \mathcal{H} \to \mathcal{F}_{\Gamma,\mathcal{C}}(B)$ the limit and the map in the statement respectively. The category $J_{B,\mathcal{C}}$ is not empty because $(\mathcal{E}, 0) \in J_{B,\mathcal{C}}$ for all $\mathcal{E} \in \mathcal{C}$ and the map α is well defined because, for all $x \in \Gamma_{\overline{\mathcal{E}}}$ and for all $u \colon (\mathcal{E}, \psi) \to (\overline{\mathcal{E}}, u(\psi))$ we have

$x_{\overline{\mathcal{E}},u(\psi)} = (\Gamma_u(x))_{\mathcal{E},\psi}$, by definition of $\mathcal{F}_{\Gamma,\mathcal{C}}(B)$ as coend. Moreover if $(\mathcal{E}_1,\psi_1),(\mathcal{E}_2,\psi_2) \in J_{B,\mathcal{C}}$ then we have maps $\mathrm{pr}_i\colon (\mathcal{E}_1\oplus\mathcal{E}_2,\psi_1\oplus\psi_2) \to (\mathcal{E}_i,\psi_i)$ for $i=1,2$, where pr_i is the projection.

Given an A-module N then $\mathrm{Hom}_A(\mathcal{H},N)$ is A-linearly isomorphic to the set of natural transformations of sets $\beta_{\mathcal{E}}\colon \mathcal{E}(B) \to \mathrm{Hom}_A(\Gamma_{\mathcal{E}},N)$. Since $\mathcal{C}$ is additive, by 2.17, those transformations are automatically R-linear. So, if we denote by $F\colon \mathcal{C}\to \mathrm{Mod}(B)$ the evaluation if $\mathrm{Spec}\, B \to \mathcal{X}$, we have

$$\mathrm{Hom}_A(\mathcal{H},N) \simeq \mathrm{Hom}_{\mathrm{L}_R(C,R)}(F,\mathrm{Hom}_A(\Gamma,N)) \simeq \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},A)}(\Gamma,\mathrm{Hom}_R(F,N))$$

By 2.2 and 2.3 the last module is $\mathrm{Hom}_A(\mathcal{F}_{\Gamma,\mathcal{C}}(B),N)$. One can now check that the induced map $\mathcal{H}\to\mathcal{F}_{\Gamma,\mathcal{C}}(B)$ is the one in the statement. The last claim follows by construction: $x_{\mathcal{E},b\psi}$ is the image of $x\otimes(b\psi) = b(x\otimes\psi)$ under the map $\Gamma_{\mathcal{E}}\otimes_R\mathcal{E}(B)\to\mathcal{F}_{\Gamma,\mathcal{C}}(B)$. The last claim follows from 2.18. □

2.2. **Sheafifying R-linear monoidal functors.** In this section we show how "ring structures" on a quasi-coherent sheaf over $\mathcal{X}$ correspond to "monoidal" structures on the corresponding Yoneda functor.

We start setting up some definitions:

Definition 2.21. Let $\mathscr{C}$ and $\mathscr{D}$ be R-linear symmetric monoidal categories. A (contravariant) *pseudo-monoidal* functor $\Omega\colon\mathscr{C}\to\mathscr{D}$ is an R-linear (and contravariant) functor together with a natural transformation

$$\iota^{\Omega}_{V,W}\colon \Omega_V\otimes\Omega_W \to \Omega_{V\otimes W} \text{ for } V,W\in\mathscr{C}$$

A (contravariant) pseudo-monoidal functor $\Omega\colon\mathscr{C}\to\mathscr{D}$ is

1) *symmetric* or commutative if for all $V,W\in\mathscr{C}$ the following diagram is commutative

$$\begin{array}{ccc} \Omega_V\otimes\Omega_W & \xrightarrow{\iota^{\Omega}_{V,W}} & \Omega_{V\otimes W} \\ \downarrow & & \downarrow \\ \Omega_W\otimes\Omega_V & \xrightarrow{\iota^{\Omega}_{W,V}} & \Omega_{W\otimes V} \end{array}$$

where the vertical arrows are the obvious isomorphisms;

2) *associative* if for all $V,W,Z\in\mathscr{C}$ the following diagram is commutative

$$\begin{array}{ccc} \Omega_V\otimes\Omega_W\otimes\Omega_Z & \xrightarrow{\iota^{\Omega}_{V,W}\otimes\mathrm{id}} & \Omega_{V\otimes W}\otimes\Omega_Z \\ \downarrow{\scriptstyle \mathrm{id}\otimes\iota^{\Omega}_{W,Z}} & & \downarrow{\scriptstyle \iota^{\Omega}_{V\otimes W,Z}} \\ \Omega_V\otimes\Omega_{W\otimes Z} & \xrightarrow{\iota^{\Omega}_{V,W\otimes Z}} & \Omega_{V\otimes W\otimes Z} \end{array}$$

If I and J are the unit objects of $\mathscr{C}$ and $\mathscr{D}$ respectively, a unity for Ω is a morphism $1\colon J\to\Omega_I$ such that, for all $V\in\mathscr{C}$, the compositions

$$\Omega_V\otimes J\to\Omega_V\otimes\Omega_I\to\Omega_{V\otimes I}\to\Omega_V \text{ and } J\otimes\Omega_V\to\Omega_I\otimes\Omega_V\to\Omega_{I\otimes V}\to\Omega_V$$

coincide with the natural isomorphisms $\Omega_V\otimes J\to\Omega_V$ and $J\otimes\Omega_V\to\Omega_V$ respectively. A (contravariant) *monoidal* functor $\Omega\colon\mathscr{C}\to\mathscr{D}$ is a symmetric and associative pseudo-monoidal (contravariant) functor with a unity 1. A (contravariant) strong monoidal functor $\Omega\colon\mathscr{C}\to\mathscr{D}$ is a (contravariant) monoidal functors such that all the maps $\iota^{\Omega}_{V,W}$and $1\colon J\to\Omega_I$ are isomorphisms.

A morphism of pseudo-monoidal functors $(\Omega,\iota^{\Omega})\to(\Gamma,\iota^{\Gamma})$, called a monoidal morphism or transformation, is a natural transformation $\Omega\to\Gamma$ which commutes with the monoidal structures ι^*. A morphism of monoidal functors is a monoidal transformation preserving the unities.

Definition 2.22. We define the categories:

- $\mathrm{Rings}_A \mathcal{X}$, whose objects are $\mathscr{B} \in \mathrm{QCoh}_A \mathcal{X}$ with an A-linear map $m\colon \mathscr{B} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathscr{B} \to \mathscr{B}$, called the multiplication;
- $\mathrm{QAlg}_A \mathcal{X}$, as the (not full) subcategory of $\mathrm{Rings}_A \mathcal{X}$ whose objects are $\mathscr{B}$ with a commutative, associative multiplication with a unity and the arrows are morphisms preserving unities;

We also set $\mathrm{Rings}\,\mathcal{X} = \mathrm{Rings}_R \mathcal{X}$ and $\mathrm{QAlg}\,\mathcal{X} = \mathrm{QAlg}_R \mathcal{X}$.

Let $\mathcal{D}$ be a monoidal subcategory of $\mathrm{QCoh}\,\mathcal{X}$, that is a subcategory such that $\mathcal{O}_{\mathcal{X}} \in \mathcal{D}$ and for all $\mathcal{E}, \mathcal{E}' \in \mathcal{D}$ we have $\mathcal{E} \otimes \mathcal{E}' \in \mathcal{D}$. We define the category $\mathrm{PML}_R(\mathcal{D}, A)$ (resp. $\mathrm{ML}_R(\mathcal{D}, A)$), whose objects are $\Gamma \in \mathrm{L}_R(\mathcal{D}, A)$ with a pseudo-monoidal (resp. monoidal) structure.

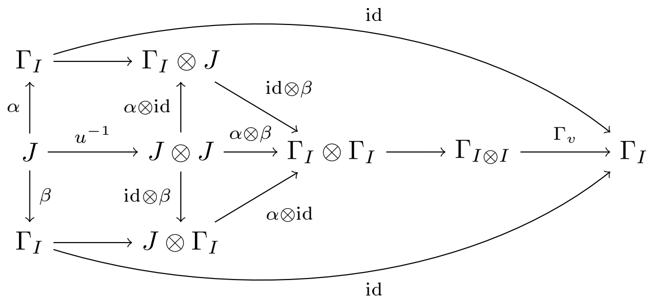
Remark 2.23. If $q\colon \mathcal{X}_A \to \mathcal{X}$ is the projection, the equivalence $\mathrm{QCoh}\,\mathcal{X}_A \to \mathrm{QCoh}_A \mathcal{X}$ extends to an equivalence

$$q_*\colon\ \mathrm{QRings}\,\mathcal{X}_A \to \mathrm{QRings}_A \mathcal{X}$$

Indeed, if $\mathcal{G} \in \mathrm{QCoh}\,\mathcal{X}_A$ then $q_*(\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{G}) = q_*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}_A}} q_*\mathcal{G}$ (see 1.19). We will use the following notation, which is somehow implicit in the definition of $\mathrm{QAlg}_A \mathcal{X}$: a sheaf $\mathscr{B} \in \mathrm{QRings}_A \mathcal{X}$ with $\mathscr{B} \simeq q_*\mathscr{B}'$ is associative (resp. commutative, has a unity, ...) if $\mathscr{B}'$ has the same property.

If $\mathscr{B} \in \mathrm{QRings}_A \mathcal{X}$ with multiplication m, then the composition $\mathscr{B} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathscr{B} \to \mathscr{B} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathscr{B} \to \mathscr{B}$ induces a ring structure on $\mathscr{B}$ as an $\mathcal{O}_{\mathcal{X}}$-module, i.e. $\mathscr{B} \in \mathrm{QRings}\,\mathcal{X}$. Moreover $\mathscr{B} \in \mathrm{QRings}_A \mathcal{X}$ is associative (resp. commutative, has a unity) if and only if $\mathscr{B} \in \mathrm{QRings}\,\mathcal{X}$ has the same property. If $\mathscr{B} \in \mathrm{QAlg}_A \mathcal{X}$ we can form the relative spectrum $\mathrm{Spec}\,\mathscr{B}$ over $\mathcal{X}_A$ and also over $\mathcal{X}$. The final result is the same.

Remark 2.24. For $\mathscr{B} \in \mathrm{Rings}_A \mathcal{X}$ or $\Gamma \in \mathrm{PMon}_R(\mathcal{D}, A)$ having a unity is a property, not an additional datum. Indeed in both cases unities are unique. For rings it is obvious. In the case of functors, using that left and right units $u\colon J \otimes J \to J$, $v\colon I \otimes I \to I$ coincides, the following diagram shows that two unities $\alpha, \beta\colon J \to \Gamma_I$ coincide.



Let $\mathcal{D}$ be a monoidal subcategory of $\mathrm{QCoh}\,\mathcal{X}$. If $\mathscr{B} \in \mathrm{Rings}_A \mathcal{X}$ with multiplication m, we endow $\Omega^{\mathscr{B}} \in \mathrm{L}_R(\mathcal{D}, A)$ with the pseudo monoidal structure

$$\iota^{\mathscr{B}}\colon\ \mathrm{Hom}(\mathcal{E}, \mathscr{B}) \otimes_A \mathrm{Hom}(\overline{\mathcal{E}}, \mathscr{B}) \to \mathrm{Hom}(\mathcal{E} \otimes \overline{\mathcal{E}}, \mathscr{B} \otimes_{\mathcal{O}_{\mathcal{X}_A}} \mathscr{B}) \to \mathrm{Hom}(\mathcal{E} \otimes \overline{\mathcal{E}}, \mathscr{B}),\ \ \mathcal{E}, \overline{\mathcal{E}} \in \mathcal{D}$$

that is $\iota^{\mathscr{B}}_{\mathcal{E},\overline{\mathcal{E}}}(\phi \otimes \psi) = m \circ (\phi \otimes \psi)$. If $1 \in \mathscr{B}$ is a unity then we set

$$A \to \Omega^{\mathscr{B}}_{\mathcal{O}_{\mathcal{X}}} = \mathrm{H}^0(\mathscr{B}),\ 1 \mapsto 1_{\mathscr{B}} = 1$$

Proposition 2.25. *The structures defined above yield an extension of the functor* $\Omega^*\colon\ \mathrm{QCoh}_A \mathcal{X} \to \mathrm{L}_R(\mathcal{D}, A)$ *to a functor* $\Omega^*\colon\ \mathrm{QRings}_A \mathcal{X} \to \mathrm{PML}_R(\mathcal{D}, A)$*. Moreover if* $\mathscr{B} \in \mathrm{Rings}_A \mathcal{X}$ *is associative (resp. commutative, has a unity* $1 \in \mathscr{B}$*) then* $\Omega^{\mathscr{B}}$ *is associative (resp. symmetric, has unity* $1_{\mathscr{B}} \in \Omega^{\mathscr{B}}_{\mathcal{O}_{\mathcal{X}}}$*). In particular we also get a functor* $\Omega^*\colon\ \mathrm{QAlg}_A \mathcal{X} \to \mathrm{ML}_R(\mathcal{D}, A)$.

Proof. Consider the expression

$$\iota^{\mathscr{B}}_{\mathcal{E},\overline{\mathcal{E}}}(\phi\otimes\psi)\colon \mathcal{E}\otimes\overline{\mathcal{E}}\ni x\otimes y\longmapsto \phi(x)\cdot\psi(y)\in\mathscr{B} \text{ for } \phi\colon\mathcal{E}\to\mathscr{B},\psi\colon\overline{\mathcal{E}}\to\mathscr{B}$$

Here $\cdot$ indicates the multiplication in $\mathscr{B}$. Associativity of $\Omega^{\mathscr{B}}$ in particular is

$$(\phi_1(x_1)\cdot\phi_2(x_2))\cdot\phi_3(x_3)=\phi_1(x_1)\cdot(\phi_2(x_2)\cdot\phi_3(x_3)) \text{ for } \phi_i\colon\mathcal{E}_i\to\mathscr{B}, x_i\in\mathcal{E}_i$$

which clearly follows if $\mathscr{B}$ is associative. Similarly $\Omega^{\mathscr{B}}$ is commutative and has a unity if

$$\phi_1(x_1)\cdot\phi_2(x_2)=\phi_2(x_2)\cdot\phi_1(x_1) \text{ and } \phi_1(x_1)\cdot 1=1\cdot\phi_1(x_1)=\phi_1(x_1) \text{ for } \phi_i\colon\mathcal{E}_i\to\mathscr{B}, x_i\in\mathcal{E}_i$$

respectively. □

Let $\mathcal{C}$ be a small monoidal subcategory of $\operatorname{QCoh}\mathcal{X}$. Given $\Gamma\in\operatorname{PMon}_R(\mathcal{C},A)$ with monoidal structure ι, we denote $\mathscr{A}_{\Gamma,\mathcal{C}}=\mathcal{F}_{\Gamma,\mathcal{C}}$ and define the multiplication $m_\Gamma\colon\mathscr{A}_{\Gamma,\mathcal{C}}\otimes_{\mathcal{O}_{\mathcal{X}_A}}\mathscr{A}_{\Gamma,\mathcal{C}}\to\mathscr{A}_{\Gamma,\mathcal{C}}$ by

$$\mathscr{A}_{\Gamma,\mathcal{C}}(B)\otimes_{B\otimes_R A}\mathscr{A}_{\Gamma,\mathcal{C}}(B)\ni x_{\mathcal{E},\psi}\otimes\overline{x}_{\overline{\mathcal{E}},\overline{\psi}}\to[\iota_{\mathcal{E},\overline{\mathcal{E}}}(x\otimes\overline{x})]_{\mathcal{E}\otimes\overline{\mathcal{E}},\psi\otimes\overline{\psi}}\in\mathscr{A}_{\Gamma,\mathcal{C}}(B)$$

where $\operatorname{Spec}B\to\mathcal{X}$ is a map, $\mathcal{E},\overline{\mathcal{E}}\in\mathcal{C}$, $\psi\in\mathcal{E}(B)$, $\overline{\psi}\in\overline{\mathcal{E}}(B)$, $x\in\Gamma_{\mathcal{E}}$, $\overline{x}\in\Gamma_{\overline{\mathcal{E}}}$ (see 2.5). In other words we claim that the sum over all $\mathcal{E},\overline{\mathcal{E}}\in\mathcal{C}$ of the compositions

$$\Gamma_{\mathcal{E}}\otimes\mathcal{E}\otimes\Gamma_{\overline{\mathcal{E}}}\otimes\overline{\mathcal{E}}\to\Gamma_{\mathcal{E}}\otimes\Gamma_{\overline{\mathcal{E}}}\otimes(\mathcal{E}\otimes\overline{\mathcal{E}})\to\Gamma_{\mathcal{E}\otimes\overline{\mathcal{E}}}\otimes(\mathcal{E}\otimes\overline{\mathcal{E}})\to\mathscr{A}_{\Gamma,\mathcal{C}}$$

induces a map $m_\Gamma\colon\mathscr{A}_{\Gamma,\mathcal{C}}\otimes_{\mathcal{O}_{\mathcal{X}_A}}\mathscr{A}_{\Gamma,\mathcal{C}}\to\mathscr{A}_{\Gamma,\mathcal{C}}$. We continue to denote by $\mathscr{A}_{\Gamma,\mathcal{C}}$ the sheaf $\mathcal{F}_{\Gamma,\mathcal{C}}$ together with the multiplication map just defined. If $1\in\Gamma_{\mathcal{O}_\mathcal{X}}$ is a unity we set $1_\Gamma\in\mathscr{A}_{\Gamma,\mathcal{C}}$ the image of 1 under the morphism $\Gamma_{\mathcal{O}_\mathcal{X}}\to\Omega^{\mathscr{A}_{\Gamma,\mathcal{C}}}_{\mathcal{O}_\mathcal{X}}=\mathrm{H}^0(\mathscr{A}_{\Gamma,\mathcal{C}})$.

Proposition 2.26. *The structures defined above yield an extension of the functor* $\mathcal{F}_{*,\mathcal{C}}\colon\mathrm{L}_R(\mathcal{C},A)\to\operatorname{QCoh}_A\mathcal{X}$ *to a functor* $\mathscr{A}_{*,\mathcal{C}}\colon\operatorname{PML}_R(\mathcal{C},A)\to\operatorname{Rings}_A\mathcal{X}$ *which is left adjoint to* $\Omega^*\colon\operatorname{Rings}_A\mathcal{X}\to\operatorname{PML}_R(\mathcal{C},A)$. *More precisely, if* $\mathscr{B}\in\operatorname{Rings}_A\mathcal{X}$ *then the morphism* $\delta_{\mathscr{B}}\colon\mathscr{A}_{\Omega^{\mathscr{B}},\mathcal{C}}\to\mathscr{B}$ *preserves multiplications and unities, while if* $\Gamma\in\operatorname{PMon}_R(\mathcal{C},A)$ *then the natural transformation* $\gamma_\Gamma\colon\Gamma\to\Omega^{\mathscr{A}_{\Gamma,\mathcal{C}}}$ *is monoidal and preserves unities (see 2.5).*

If $\Gamma\in\operatorname{PMon}_R(\mathcal{C},A)$ *is associative (resp. symmetric, has a unity* $1\in\Gamma_{\mathcal{O}_\mathcal{X}}$*) then* $\mathscr{A}_{\Gamma,\mathcal{C}}$ *is associative (resp. commutative, has unity* $1_\Gamma\in\mathscr{A}_{\Gamma,\mathcal{C}}$*). In particular we get a functor* $\mathscr{A}_{*,\mathcal{C}}\colon\operatorname{ML}_R(\mathcal{C},A)\to\operatorname{QAlg}_A\mathcal{X}$ *which is left adjoint to* $\Omega^*\colon\operatorname{QAlg}_A\mathcal{X}\to\operatorname{ML}_R(\mathcal{C},A)$.

Proof. We first show that the multiplication is well defined. Using twice the universal property of the coend definition of $\mathscr{A}_{\Gamma,\mathcal{C}}(B)$ we have that

$$\operatorname{Hom}_{B\otimes_R A}(\mathscr{A}_{\Gamma,\mathcal{C}}(B),\operatorname{End}_{B\otimes_R A}(\mathscr{A}_{\Gamma,\mathcal{C}}(B)))$$

equals

$$\operatorname{Hom}_{\mathrm{L}_R(\mathcal{C},A)}(\Gamma,\operatorname{Hom}_B(F,\operatorname{Hom}_{\mathrm{L}_R(\mathcal{C},A)}(\Gamma,\operatorname{Hom}_B(F,\mathscr{A}_{\Gamma,\mathcal{C}}(B)))))$$

An object in the above set can be seen as a map

$$\Gamma_{\mathcal{E}}\times\mathcal{E}(B)\times\Gamma_{\overline{\mathcal{E}}}\times\overline{\mathcal{E}}(B)\to\mathscr{A}_{\Gamma,\mathcal{C}}$$

which is A-linear in the first and third coordinates and B-linear in the others. Moreover there is a compatibility with respect to morphisms in the variables $\mathcal{E}$ and $\overline{\mathcal{E}}$. Thus everything boils down in proving these properties for the map

$$(x,\psi,\overline{x},\overline{\psi})\mapsto[\iota_{\mathcal{E},\overline{\mathcal{E}}}(x\otimes\overline{x})]_{\mathcal{E}\otimes\overline{\mathcal{E}},\psi\otimes\overline{\psi}}$$

This is elementary and left to the reader. Also the fact that if Γ is a associative (resp. symmetric, has a unity $1\in\Gamma_{\mathcal{O}_\mathcal{X}}$) then $\mathscr{A}_{\Gamma,\mathcal{C}}$ is associative (resp. commutative, has unity $1_\Gamma\in\mathscr{A}_{\Gamma,\mathcal{C}}$) is a tedious elementary proof.

The fact that $\gamma_\Gamma\colon\Gamma\to\Omega^{\mathscr{A}_{\Gamma,\mathcal{C}}}$ is monoidal and preserve the unit really comes from construction. Same for the proof that $\delta_{\mathscr{B}}\colon\mathscr{A}_{\Omega^{\mathscr{B}},\mathcal{C}}\to\mathscr{B}$ preserves multiplications and unities. The adjointness of the functor in the statement follows formally from this. □

3. Yoneda embeddings.

In this section we address the problem of when the Yoneda functor $\mathrm{QCoh}_A \mathcal{X} \to \mathrm{L}_R(\mathcal{C}, A)$ is fully faithful and describe its essential image. This will led us to the notion of generating category and left exactness for functors in $\mathrm{L}_R(\mathcal{C}, A)$.

We fix an R-algebra A, a pseudo-geometric fibered category $\mathcal{X}$ and a small subcategory $\mathcal{C}$ of $\mathrm{QCoh}\,\mathcal{X}$. In particular $\mathrm{QCoh}(\mathcal{X})$ is an abelian category.

Definition 3.1. Let $\mathcal{D} \subseteq \mathrm{QCoh}\,\mathcal{X}$ be a subcategory. A sheaf $\mathcal{G} \in \mathrm{QCoh}(\mathcal{X})$ is generated by $\mathcal{D}$ if there exists a surjective morphism

$$\bigoplus_{i\in I} \mathcal{E}_i \to \mathcal{G}$$

where I is a set and $\mathcal{E}_i \in \mathcal{D}$ for all $i \in I$. A sheaf $\mathcal{G} \in \mathrm{QCoh}_A \mathcal{X}$ is generated by $\mathcal{D}$ if it is so as an object of $\mathrm{QCoh}\,\mathcal{X}$. Equivalently, a sheaf $\mathcal{G} \in \mathrm{QCoh}\,\mathcal{X}_A$ is generated by $\mathcal{D}$ if $h_*\mathcal{G} \in \mathrm{QCoh}\,\mathcal{X}$ is so, where $h\colon \mathcal{X}_A \to \mathcal{X}$ is the projection. We define $\mathrm{QCoh}^{\mathcal{D}}_A \mathcal{X}$ as the subcategory of $\mathrm{QCoh}_A \mathcal{X}$ of sheaves $\mathcal{G}$ generated by $\mathcal{D}$ and such that, for all maps $\mathcal{E} \to \mathcal{G}$ with $\mathcal{E} \in \mathcal{D}^{\oplus}$, also $\mathrm{Ker}\,\psi$ is generated by $\mathcal{D}$.

If $\mathcal{D}'$ is another subcategory of $\mathrm{QCoh}\,\mathcal{X}$ we will say that $\mathcal{D}$ generates $\mathcal{D}'$ if all quasi-coherent sheaves in $\mathcal{D}'$ are generated by $\mathcal{D}$.

Remark 3.2. Consider a set of morphisms $\{U_j = \mathrm{Spec}\,B_j \to \mathcal{X}\}_{j\in J}$ such that $\sqcup_j U_j \to \mathcal{X}$ is an atlas. By 1.8 we have the following characterizations. If $\mathcal{G} \in \mathrm{QCoh}\,\mathcal{X}$ then $\mathcal{G}$ is generated by $\mathcal{D}$ if and only if

$$\forall j \in J,\ x \in \mathcal{G}(B_j),\ \exists \mathcal{E} \in \mathcal{D}^{\oplus},\ \phi \in \mathcal{E}(B_j),\ u\colon \mathcal{E} \to \mathcal{G} \text{ such that } u(\phi) = x$$

If $\mathcal{H} \in \mathrm{QCoh}(\mathcal{X})$ and $\psi\colon \mathcal{H} \to \mathcal{G}$ is a map then $\mathrm{Ker}\,\psi$ is generated by $\mathcal{D}$ if and only

$$\forall j \in J,\ y \in \mathcal{H}(B_j) \text{ with } \psi(y) = 0,\ \exists\, \overline{\mathcal{E}} \in \mathcal{D}^{\oplus},\ \phi \in \overline{\mathcal{E}}(B_j),\ v\colon \overline{\mathcal{E}} \to \mathcal{H} \text{ such that } \psi v = 0,\ v(\phi) = y$$

In particular if $\mathcal{G} \in \mathrm{QCoh}^{\mathcal{D}}\,\mathcal{X}$, $\mathcal{H} \in \mathrm{QCoh}\,\mathcal{X}$ is generated by $\mathcal{D}$ and $\mathcal{H} \to \mathcal{G}$ is a map then $\mathrm{Ker}\,\alpha$ is generated by $\mathcal{D}$.

Proposition 3.3. *If $\mathcal{D}$ is a subcategory of $\mathrm{QCoh}\,\mathcal{X}$ then the category $\mathrm{QCoh}^{\mathcal{D}}_A \mathcal{X}$ is stable by direct sums. In particular $\mathcal{D} \subseteq \mathrm{QCoh}^{\mathcal{D}}\,\mathcal{X} \iff \mathcal{D}^{\oplus} \subseteq \mathrm{QCoh}^{\mathcal{D}}\,\mathcal{X}$.*

Proof. Let $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}^{\mathcal{D}}_A \mathcal{X}$. Clearly $\mathcal{F}\oplus\mathcal{G}$ is generated by $\mathcal{D}$. Now consider a map $\alpha\colon \mathcal{E} \to \mathcal{F}\oplus\mathcal{G}$ with $\mathcal{E} \in \mathcal{D}^{\oplus}$ and write $\alpha = \phi \oplus \psi$. By 3.2 it follows that $\mathrm{Ker}\,\alpha = \mathrm{Ker}(\phi_{|\,\mathrm{Ker}(\psi)}\colon \mathrm{Ker}\,\psi \to \mathcal{F})$ is generated by $\mathcal{D}$ because $\mathrm{Ker}(\psi)$ is generated by $\mathcal{D}$ and $\mathcal{F} \in \mathrm{QCoh}^{\mathcal{D}}_A \mathcal{X}$. □

If $g\colon U = \mathrm{Spec}\,B \to \mathcal{X}$ is a map from a scheme then $(J_{B,\mathcal{C}})^{\mathrm{op}}$ (see 2.20) is not a filtered category in general. Thus if $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ and $x_{\mathcal{E},\phi} \in \mathcal{F}_{\Gamma,\mathcal{C}}(B)$ it is very difficult to understand when $x_{\mathcal{E},\phi}$ is zero in $\mathcal{F}_{\Gamma,\mathcal{C}}(B)$. Luckily, under some hypothesis this is possible.

Lemma 3.4. *Assume $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}}\,\mathcal{X}$. Then for all flat maps $g\colon \mathrm{Spec}\,B \to \mathcal{X}$ the category $(J_{B,\mathcal{C}^{\oplus}})^{op}$ is filtered. In this case, given $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$, every element of $\mathcal{F}_{\Gamma,\mathcal{C}}(B)$ are of the form $x_{\mathcal{E},\phi}$ for some $(\mathcal{E}, \phi) \in J_{B,\mathcal{C}}$ and such an element is zero if and only if there exists $(\overline{\mathcal{E}}, \overline{\phi}) \to (\mathcal{E}, \phi)$ in $J_{B,\mathcal{C}}$ such that $\Gamma_u(x) = 0$.*

Proof. By 2.16 and 3.3 we can assume $\mathcal{C} = \mathcal{C}^{\oplus}$. The last claim follows from 2.20 and the fact that $J^{\mathrm{op}}_{B,\mathcal{C}}$ is filtered, which we now prove. By 2.20 it remains to show that for all maps $\alpha, \beta\colon (\mathcal{E}, \phi) \to (\overline{\mathcal{E}}, \overline{\phi})$ in $J_{B,\mathcal{C}}$ there exists $u\colon (\mathcal{E}', \phi') \to (\mathcal{E}, \phi)$ in $J_{B,\mathcal{C}}$ such that $\alpha u = \beta u$. Let $\mathcal{K} = \mathrm{Ker}((\alpha - \beta)\colon \mathcal{E} \to \overline{\mathcal{E}})$. Since $\mathrm{Spec}\,B \to \mathcal{X}$ is flat, by 1.13 one has

$$\phi \in \mathcal{K}(B) = \mathrm{Ker}((\alpha_B - \beta_B)\colon \mathcal{E}(B) \to \overline{\mathcal{E}}(B))$$

By assumption $\mathcal{K}$ is generated by $\mathcal{C}$ and, since $\mathcal{C}$ is additive, there exist $\mathcal{E}' \in \mathcal{C}$, a map $u\colon \mathcal{E}' \to \mathcal{K}$ and $\phi' \in \mathcal{E}'(B)$ such that $u(\phi') = \phi$. So $(\mathcal{E}', \phi') \to (\mathcal{E}, \phi)$ is an arrow in $J_{B,\mathcal{C}}$ such that $(\alpha - \beta)u = 0$ as required. □

In what follows we work out sufficient (and sometimes necessary) conditions for the surjectivity or injectivity of $\delta_\mathcal{G}\colon \mathcal{F}_{\Omega^\mathcal{G},\mathcal{C}} \to \mathcal{G}$. Recall that $\delta_\mathcal{G}(u_{\mathcal{E},\phi}) = u(\phi)$ for $\mathcal{E} \in \mathcal{C}^\oplus$, $\operatorname{Spec} B \to \mathcal{X}$, $\phi \in \mathcal{E}(B)$, $u \in \Omega^\mathcal{G}_\mathcal{E} = \operatorname{Hom}_\mathcal{X}(\mathcal{E}, \mathcal{G})$ (see 2.6).

Lemma 3.5. *If $\mathcal{G} \in \operatorname{QCoh}_A \mathcal{X}$ the map $\delta_\mathcal{G}\colon \mathcal{F}_{\Omega^\mathcal{G},\mathcal{C}} \to \mathcal{G}$ is surjective if and only if $\mathcal{G}$ is generated by $\mathcal{C}$.*

Proof. By 2.16 we can assume $\mathcal{C} = \mathcal{C}^\oplus$. Let $\{g_i\colon U_i = \operatorname{Spec} B_i \to \mathcal{X}\}$ be a set of maps such that $\sqcup_i U_i \to \mathcal{X}$ is an atlas. By 1.8 $\delta_\mathcal{G}$ is surjective if and only if $\delta_{\mathcal{G},U_i}\colon \mathcal{F}_{\Omega^\mathcal{G},\mathcal{C}}(U_i) \to \mathcal{G}(U_i)$ is surjective for all $i \in I$. By 2.20 $\operatorname{Im} \delta_{\mathcal{G},U_i}$ is the set of elements of $\mathcal{G}(U_i)$ of the form $\delta_\mathcal{G}(u_{\mathcal{E},\phi}) = u(\phi)$ for $\mathcal{E} \in \mathcal{C}^\oplus$, $\phi \in \mathcal{E}(U_i)$, $u \in \Omega^\mathcal{G}_\mathcal{E} = \operatorname{Hom}_\mathcal{X}(\mathcal{E}, \mathcal{G})$. So the claim follows from 3.2. □

Lemma 3.6. *Let $\mathcal{G} \in \operatorname{QCoh}_A \mathcal{X}$. If for all maps $\mathcal{E} \to \mathcal{G}$ with $\mathcal{E} \in \mathcal{C}^\oplus$ the kernel $\operatorname{Ker} \phi$ is generated by $\mathcal{C}$ then the map $\delta_\mathcal{G}\colon \mathcal{F}_{\Omega^\mathcal{G},\mathcal{C}} \to \mathcal{G}$ is injective. The converse holds if $\mathcal{C} \subseteq \operatorname{QCoh}^\mathcal{C} \mathcal{X}$.*

Proof. By 2.16 we can assume $\mathcal{C} = \mathcal{C}^\oplus$. Let $\{g_i\colon U_i = \operatorname{Spec} B_i \to \mathcal{X}\}$ be a set of maps such that $\sqcup_i U_i \to \mathcal{X}$ is an atlas. By 1.8 $\delta_\mathcal{G}$ is injective if and only if $\delta_{\mathcal{G},U_i}\colon \mathcal{F}_{\Omega^\mathcal{G},\mathcal{C}}(U_i) \to \mathcal{G}(U_i)$ is injective for all $i \in I$. We start proving that $\delta_\mathcal{G}$ is injective if the hypothesis in the statement are fulfilled. So let $z \in \operatorname{Ker} \delta_{\mathcal{G},U_i}$. By 2.20 there exists $\mathcal{E} \in \mathcal{C}$, $\phi \in \mathcal{E}(U_i)$ and $u\colon \mathcal{E} \to \mathcal{G}$ such that $z = u_{\mathcal{E},\phi}$. Moreover $\delta_\mathcal{G}(u_{\mathcal{E},\phi}) = u(\phi) = 0$. Set $\mathcal{K} = \operatorname{Ker} u$. Since $\phi \in \mathcal{K}(U_i)$ and, by hypothesis, $\mathcal{K}$ is generated by $\mathcal{C}$ there exist $\overline{\mathcal{E}} \in \mathcal{C}$, $\overline{\phi} \in \overline{\mathcal{E}}(U_i)$ and a map $v\colon \overline{\mathcal{E}} \to \mathcal{K}$ such that $v(\overline{\phi}) = \phi$. If we denote by v also the composition $\overline{\mathcal{E}} \to \mathcal{K} \to \mathcal{E}$ we have

$$u_{\mathcal{E},\phi} = (\Omega^\mathcal{G}_v(u))_{\overline{\mathcal{E}},\overline{\phi}} = (uv)_{\overline{\mathcal{E}},\overline{\phi}} = 0_{\overline{\mathcal{E}},\overline{\phi}} = 0 \text{ in } \mathcal{F}_{\Gamma,\mathcal{C}}(U_i)$$

Assume now that $\delta_\mathcal{G}$ is injective and $\mathcal{C} \subseteq \operatorname{QCoh}^\mathcal{C} \mathcal{X}$ and let $u\colon \mathcal{E} \to \mathcal{G}$ be a map with $\mathcal{E} \in \mathcal{C}$. We have to prove that $\mathcal{K} = \operatorname{Ker} u$ is generated by $\mathcal{C}$. If $\phi \in \mathcal{K}(U_i) \subseteq \mathcal{E}(U_i)$, then $u(\phi) = \delta_\mathcal{G}(u_{\mathcal{E},\phi}) = 0$. So $u_{\mathcal{E},\phi} = 0$ and the conclusion follows from 3.2 and 3.4. □

In general we can still conclude that:

Proposition 3.7. *If $\mathcal{E} \in \mathcal{C}^\oplus$ then the map $\delta_\mathcal{E}\colon \mathcal{F}_{\Omega^\mathcal{E},\mathcal{C}} \to \mathcal{E}$ is an isomorphism.*

Proof. By 2.16 we can assume $\mathcal{C} = \mathcal{C}^\oplus$. Let $\mathcal{H} \in \operatorname{QCoh} \mathcal{X}$. The map

$$\operatorname{Hom}_\mathcal{X}(\mathcal{E}, \mathcal{H}) \to \operatorname{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\Omega^\mathcal{E}, \Omega^\mathcal{H}) \simeq \operatorname{Hom}_\mathcal{X}(\mathcal{F}_{\Omega^\mathcal{E},\mathcal{C}}, \mathcal{H})$$

maps $\mathrm{id}_\mathcal{E}$ to $\delta_\mathcal{E}$ and thus is induced by $\delta_\mathcal{E}$. By the enriched Yoneda's lemma or a direct check we see that the above map and therefore $\delta_\mathcal{E}$ are isomorphisms. □

Theorem 3.8. *Let $\mathcal{D}_A$ be the subcategory of $\operatorname{QCoh}_A \mathcal{X}$ of sheaves $\mathcal{G}$ such that $\delta_\mathcal{G}\colon \mathcal{F}_{\Omega^\mathcal{G},\mathcal{C}} \to \mathcal{G}$ is an isomorphism. Then $\mathcal{D}_A$ is an additive category containing $\operatorname{QCoh}^\mathcal{C}_A \mathcal{X}$, $\mathcal{C}^\oplus \subseteq \mathcal{D}_R$ and the functor*

$$\Omega^*\colon \mathcal{D}_A \to \mathrm{L}_R(\mathcal{C}, A)$$

is fully faithful. Moreover if $\mathcal{C} \subseteq \operatorname{QCoh}^\mathcal{C} \mathcal{X}$ then $\mathcal{D}_A = \operatorname{QCoh}^\mathcal{C}_A \mathcal{X}$.

Proof. The category $\mathcal{D}_A$ is additive because Ω^* and $\mathcal{F}_{*,\mathcal{C}}$ are additive. All the other claims follows from 3.5, 3.6, 3.7 and the fact that $\delta_\mathcal{G}$ is the counit of an adjunction. □

4. Left exactness

We keep the notation from the previous section. So $\mathcal{X}$ is a pseudo-algebraic fibered category over a ring R, $\mathcal{C} \subseteq \mathrm{QCoh}(\mathcal{X})$ is a small full subcategory and $\mathcal{D} \subseteq \mathrm{QCoh}(\mathcal{X})$ is any full subcategory. The symbol A instead will always denote an R-algebra.

Now we want to address the problem of what is the essential image of the Yoneda functor $\Omega^*\colon \mathrm{QCoh}_A \mathcal{X} \to \mathrm{L}_R(\mathcal{D}, A)$. We will see that for $\mathcal{F} \in \mathrm{QCoh}_A \mathcal{X}$ the associated Yoneda functor $\Omega^{\mathcal{F}}$ is always "left exact" and we will give sufficient conditions assuring that "left exact" functors in $\mathrm{L}_R(\mathcal{D}, A)$ are Yoneda functors associated with some quasi-coherent sheaf on $\mathcal{X}$. Since $\mathcal{D}$ is not abelian, we introduce an ad hoc notion of left exactness.

Definition 4.1. Let $\mathcal{D}$ be a subcategory of $\mathrm{QCoh}\,\mathcal{X}$. A map

$$\alpha = (\alpha_{kj})\colon \bigoplus_{k\in K} \mathcal{E}_k \to \bigoplus_{j\in J} \mathcal{E}_j \text{ with } \mathcal{E}_k, \mathcal{E}_j \in \mathcal{D}$$

where K and J are sets, is called locally finite (with respect to the decomposition) if all the maps $\mathcal{E}_k \to \bigoplus_j \mathcal{E}_j$ factors through a finite sum. If $\Gamma \in \mathrm{L}_R(\mathcal{D}, A)$ we set

$$\Gamma_\alpha\colon \prod_{j\in J} \Gamma_{\mathcal{E}_j} \to \prod_k \Gamma_{\mathcal{E}_k},\ x = (x_j)_j \mapsto (\sum_j \Gamma_{\alpha_{kj}}(x_j))_k$$

Remark 4.2. It is easy to verify that the above association is functorial, that is $\Gamma_{\beta\circ\alpha} = \Gamma_\alpha \circ \Gamma_\beta$ when it makes sense. If K and J are finite then Γ_α is obtained applying the extension $\Gamma \in \mathrm{L}_R(\mathcal{D}^\oplus, A)$ (see 2.16) on the map α.

We should warn the reader that even if α is an arrow in $\mathcal{D}$ the map Γ_α in definition above is not obtained applying Γ on the arrow. The problem here is that Γ may not transform direct sums into direct products. This is clearly an abuse of notation, but we hope it will not led to confusion. The map α as well as the notion of local finiteness should be interpreted in a category of "decompositions".

Definition 4.3. A *test sequence* for $\mathcal{D}$ is an exact sequence

$$\bigoplus_{k\in K} \mathcal{E}_k \to \bigoplus_{j\in J} \mathcal{E}_j \to \mathcal{E} \to 0 \text{ with } \mathcal{E}, \mathcal{E}_j, \mathcal{E}_k \in \mathcal{D} \text{ for all } j \in J, k \in K \tag{4.1}$$

in $\mathrm{QCoh}\,\mathcal{X}$ where the first map is locally finite. We will also say that it is a test sequence for $\mathcal{E} \in \mathcal{D}$. A finite test sequence for $\mathcal{E} \in \mathcal{D}$ is an exact sequence

$$\mathcal{E}'' \to \mathcal{E}' \to \mathcal{E} \to 0 \text{ with } \mathcal{E}', \mathcal{E}'' \in \mathcal{D}^\oplus$$

Given $\Gamma \in \mathrm{L}_R(\mathcal{D}, A)$ we say that Γ is exact on the test sequence (4.1) if the sequence

$$\begin{array}{ccccccc} & & & & (x_j)_j & \longmapsto & (\sum_j \Gamma_{u_{kj}}(x_j))_k \\ 0 & \longrightarrow & \Gamma_{\mathcal{E}} & \longrightarrow & \prod_{j\in J} \Gamma_{\mathcal{E}_j} & \longrightarrow & \prod_{k\in K} \Gamma_{\mathcal{E}_k} \\ & & x & \longmapsto & (\Gamma_{u_j}(x))_j & & \end{array} \tag{4.2}$$

is exact, where $u_j\colon \mathcal{E}_j \to \mathcal{E}$, $u_{kj}\colon \mathcal{E}_k \to \mathcal{E}_j$ denotes the maps in (4.1). We say that Γ is left exact if it is exact on all test sequences in $\mathcal{D}$. We define $\mathrm{Lex}_R(\mathcal{D}, A)$ as the subcategory of $\mathrm{L}_R(\mathcal{D}, A)$ of left exact functors.

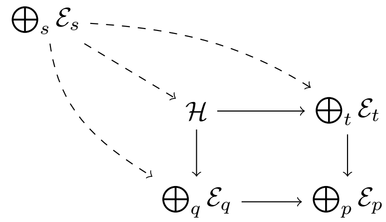
Remark 4.4. If $\mathcal{D} \subseteq \mathrm{QCoh}^{\mathcal{D}}\,\mathcal{X}$ then any surjective map $\mu\colon \bigoplus_{j\in J} \mathcal{E}_j \to \mathcal{E} \to 0$ can be extended to a test sequence in $\mathcal{C}$. More generally if $u\colon \bigoplus_{j\in J} \mathcal{E}_j \to \bigoplus_{p\in P} \mathcal{E}_p$ is a locally finite map, then there exists a locally finite map $\bigoplus_k \mathcal{E}_k \to \bigoplus_j \mathcal{E}_j$ whose image is $\mathrm{Ker}(u)$. Indeed let Q be the set of finite subsets of J and for any $T \in Q$ let $P_T \subseteq P$ be a finite subsets such that each $\mathcal{E}_j$ for $j \in T$ maps inside $\bigoplus_{p \in P_T} \mathcal{E}_p$. By 3.3 $\mathcal{D}^{\oplus} \subseteq \mathrm{QCoh}^{\mathcal{D}} \mathcal{X}$ and therefore we can find exact sequences

$$\bigoplus_{k \in K_T} \mathcal{E}_k \to \bigoplus_{j \in T} \mathcal{E}_j \to \bigoplus_{p \in P_T} \mathcal{E}_p$$

Since u is locally finite we have an exact sequence

$$\bigoplus_{T \in Q} \bigoplus_{k \in K_T} \mathcal{E}_k \to \bigoplus_{j \in J} \mathcal{E}_j \to \bigoplus_{p \in P} \mathcal{E}_p$$

where the first map is locally finite. In particular given a Cartesian diagram of solid arrows



in which all the maps are locally finite then there exists a surjective map $\bigoplus_s \mathcal{E}_s \to \mathcal{H}$ such that the other dashed arrows are locally finite. This is because $\mathcal{H}$ is the kernel of the difference map

$$(\bigoplus_q \mathcal{E}_q) \bigoplus (\bigoplus_t \mathcal{E}_t) \to \bigoplus_p \mathcal{E}_p$$

which is locally finite.

Proposition 4.5. *If $\mathcal{F} \in \mathrm{QCoh}_A \mathcal{X}$ then $\Omega^{\mathcal{F}} \in \mathrm{Lex}_R(\mathcal{D}, A)$. More generally $\Omega^{\mathcal{F}}$ is exact on all exact sequences of the form*

$$\bigoplus_{j \in J} \mathcal{E}_j \to \bigoplus_{q \in Q} \mathcal{E}_q \to 0 \text{ or } \bigoplus_{k \in K} \mathcal{E}_k \to \bigoplus_{j \in J} \mathcal{E}_j \to \bigoplus_{q \in Q} \mathcal{E}_q \to 0 \text{ for } \mathcal{E}_k, \mathcal{E}_j, \mathcal{E}_q \in \mathcal{C}$$

where the maps involved are locally finite.

Proof. It is enough to observe that $\mathrm{Hom}_{\mathcal{X}}(-, \mathcal{F})$ is left exact in the usual sense and that

$$\mathrm{Hom}_{\mathcal{X}}(\bigoplus_i \mathcal{E}_i, \mathcal{F}) \simeq \prod_i \mathrm{Hom}(\mathcal{E}_i, \mathcal{F}) \simeq \prod_i \Omega^{\mathcal{F}}_{\mathcal{E}_i}$$

□

Proposition 4.6. *The functor $\Omega^* \colon \mathrm{QCoh}_A \mathcal{X} \to \mathrm{L}_R(\mathcal{D}, A)$ is left exact. If $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ then $\mathcal{F}_{*,\mathcal{C}} \colon \mathrm{L}_R(\mathcal{C}, A) \to \mathrm{QCoh}_A \mathcal{X}$ is exact.*

Proof. For the first claim it is enough to use that $\mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, -)$ is left exact. For the last part of the statement consider a set of maps $\{U_i = \operatorname{Spec} B_i \to \mathcal{X}\}_{i \in I}$ such that $\sqcup_i U_i \to \mathcal{X}$ is an atlas. Let also $\Gamma' \to \Gamma \to \Gamma''$ be an exact sequence in $\mathrm{L}_R(\mathcal{C}, A)$. By 2.20 the sequence $\mathcal{F}_{\Gamma',\mathcal{C}}(B_i) \to \mathcal{F}_{\Gamma,\mathcal{C}}(B_i) \to \mathcal{F}_{\Gamma'',\mathcal{C}}(B_i)$ are exact for all $i \in I$ because limit of exact sequences $\Gamma'_{\mathcal{E},\phi} \to \Gamma_{\mathcal{E},\phi} \to \Gamma''_{\mathcal{E},\phi}$ over the category $(J_{B_i,\mathcal{C}})^{\mathrm{op}}$, which is filtered thanks to 3.4. Applying 1.8 we get the result. □

Recall that if $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ then $\gamma_{\Gamma,\mathcal{E}} \colon \Gamma_{\mathcal{E}} \to \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}_{\mathcal{E}} = \mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}_{\Gamma,\mathcal{C}})$ is given by $\gamma_{\Gamma,\mathcal{E}}(x)(\phi) = x_{\mathcal{E},\phi}$ for $\mathcal{E} \in \mathcal{C}$, $x \in \Gamma_{\mathcal{E}}$, $\operatorname{Spec} B \to \mathcal{X}$ and $\phi \in \mathcal{E}(B)$ (see 2.6).

Lemma 4.7. *Assume $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$. If $\Gamma \in \mathrm{Lex}_R(\mathcal{C}, A)$ and the map $\Omega^* \circ \mathcal{F}_{*,\mathcal{C}}(\gamma_\Gamma) \colon \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}} \to \Omega^{\mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}}}$ is an isomorphism then the natural transformation $\gamma_\Gamma \colon \Gamma \to \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}$ is an isomorphism.*

Proof. Let $\{U_i = \operatorname{Spec} B_i \to \mathcal{X}\}_{i\in I}$ be a set of maps such that $\sqcup_i U_i \to \mathcal{X}$ is an atlas and let $\Psi \in \mathrm{L}_R(\mathcal{C}, A)$ and $x \in \operatorname{Ker} \gamma_{\Psi,\mathcal{E}}$ for some $\mathcal{E} \in \mathcal{C}$. We are going to prove that there exists a surjective map $\mu = \oplus_j \mu_j \colon \oplus_{j\in J} \mathcal{E}_j \to \mathcal{E}$ with $\mathcal{E}_j, \mathcal{E} \in \mathcal{C}$ such that $\Psi_{u_j}(x) = 0$ for all j.

If $\phi \in \mathcal{E}(U_i)$, by 3.4 and the fact that $\gamma_{\Psi,\mathcal{E}}(x)(\phi) = x_{\mathcal{E},\phi}$ is zero in $\mathcal{F}_{\Psi,\mathcal{C}}(U_i)$, there exists $(\mathcal{E}_\phi, y_\phi) \in J_{B_i,\mathcal{C}}$ and a map $u_\phi \colon (\mathcal{E}_\phi, y_\phi) \to (\mathcal{E}, \phi)$ such that $\Psi_{u_\phi}(x) = 0$. Consider the induced map

$$\bigoplus_{i\in I} \bigoplus_{\phi\in\mathcal{E}(U_i)} \mathcal{E}_\phi \to \mathcal{E}$$

which is surjective by 1.8. Writing all the $\mathcal{E}_\phi \in \mathcal{C}^\oplus$ as sums of sheaves in $\mathcal{C}$ we get the desired surjective map.

We return now to the proof of the statement. Given $x \in \operatorname{Ker} \gamma_{\Gamma,\mathcal{E}}$, considering a surjection μ as above for $\Psi = \Gamma$, extending it via 4.4 and using the exactness of Γ we can conclude that $x = 0$. This means that the natural transformation $\gamma_\Gamma \colon \Gamma \to \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}$ is injective. Set now $\Pi = \operatorname{Coker} \gamma_{\Gamma,\mathcal{C}}$. By 4.6 we have an exact sequence

$$0 \to \mathcal{F}_{\Gamma,\mathcal{C}} \to \mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}} \to \mathcal{F}_{\Pi,\mathcal{C}} \to 0$$

This is a split sequence because the composition of $\mathcal{F}_{*,\mathcal{C}}(\gamma_\Gamma) \colon \mathcal{F}_{\Gamma,\mathcal{C}} \to \mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}}$ and $\delta_{\mathcal{F}_{\Gamma,\mathcal{C}}} \colon \mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}} \to \mathcal{F}_{\Gamma,\mathcal{C}}$ is the identity. So Ω^* maintains the exactness of the above sequence and therefore

$$\Omega^{\mathcal{F}_{\Pi,\mathcal{C}}} = \operatorname{Coker}(\Omega^* \circ \mathcal{F}_{*,\mathcal{C}}(\gamma_\Gamma)) = 0$$

by hypothesis. We want to prove that $\Pi = 0$. Let $x \in \Pi_\mathcal{E}$ for $\mathcal{E} \in \mathcal{C}$. Since $\Omega^{\mathcal{F}_{\Pi,\mathcal{C}}} = 0$ we have $x \in \operatorname{Ker} \gamma_{\Pi,\mathcal{E}}$. Consider a surjection $\mu = \oplus_j \mu_j$ constructed as above starting from $x \in \Pi_\mathcal{E}$ and $\Psi = \Pi$. By 4.4 μ can be extended to a test sequence $\oplus_k \mathcal{E}_k \to \oplus_j \mathcal{E}_i \to \mathcal{E}$ because $\mathcal{C} \subseteq \operatorname{QCoh}^\mathcal{C} \mathcal{X}$. Since $\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}} \in \operatorname{Lex}_R(\mathcal{C}, A)$ by 4.5 we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Gamma_\mathcal{E} & \longrightarrow & \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}_\mathcal{E} & \longrightarrow & \Pi_\mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & \prod_{j\in J} \Gamma_{\mathcal{E}_j} & \longrightarrow & \prod_{j\in J} \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}_{\mathcal{E}_j} & \longrightarrow & \prod_{j\in J} \Pi_{\mathcal{E}_j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \prod_{k\in K} \Gamma_{\mathcal{E}_k} & \longrightarrow & \prod_{k\in K} \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}_{\mathcal{E}_k} & & & &
\end{array}$$

in which all the rows and the first two columns are exact. By diagram chasing it is easy to conclude that β is injective. Since by construction $\beta(x) = 0$ we can conclude that $x = 0$. □

Definition 4.8. We define $\operatorname{Lex}^\mathcal{C}_R(A)$ as the subcategory of $\operatorname{Lex}_R(\mathcal{C}, A)$ of functors Γ such that $\mathcal{F}_{\Gamma,\mathcal{C}} \in \operatorname{QCoh}^\mathcal{C}_A \mathcal{X}$.

Theorem 4.9. *Assume $\mathcal{C} \subseteq \operatorname{QCoh}^\mathcal{C} \mathcal{X}$. Then the functors*

$$\Omega^* \colon \operatorname{QCoh}^\mathcal{C}_A \mathcal{X} \to \operatorname{Lex}^\mathcal{C}_R(A) \text{ and } \mathcal{F}_{*,\mathcal{C}} \colon \operatorname{Lex}^\mathcal{C}_R(A) \to \operatorname{QCoh}^\mathcal{C}_A \mathcal{X}$$

are quasi-inverses of each other.

Proof. Let $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ be such that $\mathcal{F}_{\Gamma,\mathcal{C}} \in \operatorname{QCoh}^\mathcal{C}_A \mathcal{X}$. The composition

$$\delta_{\mathcal{F}_{\Gamma,\mathcal{C}}} \circ \mathcal{F}_{*,\mathcal{C}}(\gamma_\Gamma) \colon \mathcal{F}_{\Gamma,\mathcal{C}} \to \mathcal{F}_{\Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}},\mathcal{C}} \to \mathcal{F}_{\Gamma,\mathcal{C}}$$

is the identity and $\delta_{\mathcal{F}_{\Gamma,\mathcal{C}}}$ is an isomorphism since $\mathcal{F}_{\Gamma,\mathcal{C}} \in \mathrm{QCoh}_A^{\mathcal{C}} \mathcal{X}$ by 3.8. Thus $\mathcal{F}_{*,\mathcal{C}}(\gamma_\Gamma)$ and therefore $\Omega^* \circ \mathcal{F}_{*,\mathcal{C}}(\gamma_\Gamma)$ are isomorphisms. By 4.7 we can conclude that if $\Gamma \in \mathrm{Lex}_R^{\mathcal{C}}(A)$ then $\gamma_\Gamma \colon \Gamma \to \Omega^{\mathcal{F}_{\Gamma,\mathcal{C}}}$ is an isomorphism. The result then follows from 3.8 and 4.5. □

The following result allow us to extend results from small subcategories of $\mathrm{QCoh}\,\mathcal{X}$ to any subcategory.

Proposition 4.10. *The category $\mathrm{QCoh}\,\mathcal{X}$ is generated by a small subcategory. Equivalently $\mathrm{QCoh}\,\mathcal{X}$ has a generator, that is there exists $\mathcal{E} \in \mathrm{QCoh}\,\mathcal{X}$ such that $\{\mathcal{E}\}$ generates $\mathrm{QCoh}\,\mathcal{X}$.*

Proof. Follows from 1.8 and [Aut19, Tag 0780]. □

Remark 4.11. If $\mathcal{D} \subseteq \mathrm{QCoh}\,\mathcal{X}$ generates $\mathrm{QCoh}\,\mathcal{X}$ there always exists a small subcategory $\overline{\mathcal{D}} \subseteq \mathcal{D}$ that generates $\mathrm{QCoh}\,\mathcal{X}$. Indeed if $\mathcal{E}$ is a generator of $\mathrm{QCoh}\,\mathcal{X}$ it is enough to take a subset of sheaves in $\mathcal{D}$ that generates $\mathcal{E}$.

Theorem 4.12. *Let $\mathcal{D} \subseteq \mathrm{QCoh}\,\mathcal{X}$ be a subcategory that generates $\mathrm{QCoh}\,\mathcal{X}$. Then the functor*

$$\Omega^* \colon \mathrm{QCoh}_A \mathcal{X} \to \mathrm{Lex}_R(\mathcal{D}, A)$$

is an equivalence of categories and, if $\mathcal{D}$ is small, $\mathcal{F}_{*,\mathcal{D}} \colon \mathrm{Lex}_R(\mathcal{D}, A) \to \mathrm{QCoh}_A \mathcal{X}$ is a quasi-inverse. In particular if $\overline{\mathcal{D}} \subseteq \mathcal{D}$ is a subcategory that generates $\mathrm{QCoh}\,\mathcal{X}$ the restriction functor $\mathrm{Lex}_R(\mathcal{D}, A) \to \mathrm{Lex}_R(\overline{\mathcal{D}}, A)$ is an equivalence.

Proof. If $\mathcal{D}$ is small we have $\mathrm{QCoh}_A \mathcal{X} = \mathrm{QCoh}_A^{\mathcal{D}} \mathcal{X}$, $\mathrm{Lex}_R(\mathcal{D}, A) = \mathrm{Lex}_R^{\mathcal{D}}(A)$ and everything follows from 4.9. In particular the restriction $\mathrm{Lex}_R(\mathcal{D}, A) \to \mathrm{Lex}_R(\overline{\mathcal{D}}, A)$ is an equivalence if $\overline{\mathcal{D}} \subseteq \mathcal{D}$, they are small and they generate $\mathrm{QCoh}\,\mathcal{X}$. Assume now that $\mathcal{D}$ is not necessarily small and consider a small subcategory $\mathcal{C} \subseteq \mathcal{D}$ that generates $\mathrm{QCoh}\,\mathcal{X}$, which exists thanks to 4.11. The proof of the statement is complete if we prove that the restriction functor $\mathrm{Lex}_R(\mathcal{D}, A) \to \mathrm{Lex}_R(\mathcal{C}, A)$ is an equivalence. Given a set $I \subseteq \mathcal{D}$ we set $\mathcal{C}_I = \mathcal{C} \cup I \subseteq \mathcal{D}$. For all sets I, quasi-coherent sheaves are generated by $\mathcal{C}_I$. Note that we have the restriction functor $-_{|\mathcal{C}_I} \colon \mathrm{Lex}_R(\mathcal{D}, A) \to \mathrm{Lex}_R(\mathcal{C}_I, A)$ and the composition $\mathrm{QCoh}_A \mathcal{X} \to \mathrm{Lex}_R(\mathcal{D}, A) \to \mathrm{Lex}_R(C_I, A)$ is an equivalence for all I. In particular $-_{|\mathcal{C}} \colon \mathrm{Lex}_R(\mathcal{D}, A) \to \mathrm{Lex}_R(\mathcal{C}, A)$ is essentially surjective. We will conclude by proving that it is fully faithful. Let $\Gamma, \Gamma' \in \mathrm{Lex}_R(\mathcal{D}, A)$. If $\overline{I} \subseteq I$ the restriction functor $\mathrm{Lex}_R(\mathcal{C}_I, A) \to \mathrm{Lex}_R(\mathcal{C}_{\overline{I}}, A)$ is an equivalence of categories. In particular the map $\mathrm{Hom}(\Gamma_{|\mathcal{C}_I}, \Gamma'_{|\mathcal{C}_I}) \to \mathrm{Hom}(\Gamma_{|\mathcal{C}_{\overline{I}}}, \Gamma'_{|\mathcal{C}_{\overline{I}}})$ is bijective. Using this, it is elementary to prove that also the map $\mathrm{Hom}(\Gamma, \Gamma') \to \mathrm{Hom}(\Gamma_{|\mathcal{C}}, \Gamma'_{|\mathcal{C}})$ is bijective. □

5. Cohomological exactness

We keep the notation from the previous section. So $\mathcal{X}$ is a pseudo-algebraic fibered category over a ring R, $\mathcal{C} \subseteq \mathrm{QCoh}(\mathcal{X})$ is a small full subcategory and $\mathcal{D} \subseteq \mathrm{QCoh}(\mathcal{X})$ is any full subcategory. The symbol A instead will always denote an R-algebra.

In this section we want to present a cohomological interpretation of the functors in $\mathrm{Lex}_R(\mathcal{D}, A)$, which will allow us to show that it is often enough to consider just finite test sequences instead of arbitrary test sequences.

Remark 5.1. In an abelian category $\mathscr{A}$, given $X, Y \in \mathscr{A}$ we can always define the abelian group $\mathrm{Ext}^1(X, Y)$ as the group of extensions (regardless if $\mathscr{A}$ has enough injectives) and it has the usual nice properties on short exact sequences. See 010J[Aut19, Tag 010J]. In order to avoid set-theoretic problems one should require that $\mathscr{A}$ is locally small and that, given $X, Y \in \mathscr{A}$, $\mathrm{Ext}^1(X, Y)$ is a set. This is the case for $\mathscr{A} = \mathrm{L}_R(\mathcal{C}, R)$, for instance by looking at the cardinalities of the $\Gamma_{\mathcal{E}}$ for $\Gamma \in \mathrm{L}_R(\mathcal{C}, R)$ and $\mathcal{E} \in \mathcal{C}$.

Definition 5.2. Given a surjective map $\mu = \oplus_j \mu_j \colon \oplus_j \mathcal{E}_j \to \mathcal{E}$ with $\mathcal{E}, \mathcal{E}_j \in \mathcal{C}$ we set $\Omega^\mu = \oplus_j \Omega^{\mu_j} \colon \oplus_j \Omega^{\mathcal{E}_j} \to \Omega^{\mathcal{E}}$. A functor $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ is *cohomologically left exact* on μ if

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\Omega^{\mathcal{E}}/\operatorname{Im}(\Omega^\mu), \Gamma) = \mathrm{Ext}^1_{\mathrm{L}_R(\mathcal{C},R)}(\Omega^{\mathcal{E}}/\operatorname{Im}(\Omega^\mu), \Gamma) = 0 \tag{5.1}$$

It is cohomologically left exact if it is so on all surjections μ as above.

We setup some notation. We denote by $\Phi_{\mathcal{C}}(\mathcal{E})$ for $\mathcal{E} \in \mathcal{C}$ the set of subfunctor of $\Omega^{\mathcal{E}}$of the form $\operatorname{Im}(\Omega^\mu)$ for some surjective map $\mu \colon \oplus_j \mathcal{E}_i \to \mathcal{E}$ with $\mathcal{E}_j \in \mathcal{C}$ and by $\Phi_{\mathcal{C}}$ the disjoint union of all the $\Phi_{\mathcal{C}}(\mathcal{E})$: an object of $\Delta \in \Phi_{\mathcal{C}}$ is actually a pair $(\Delta, \mathcal{E})$ with $\Delta \in \Phi_{\mathcal{C}}(\mathcal{E})$. Given $\Delta = \operatorname{Im}(\Omega^\mu) \in \Phi_{\mathcal{C}}(\mathcal{E})$ and $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ we will say that Γ is cohomologically left exact on Δ if it is cohomologically left exact on μ. Notice that, using Yoneda's lemma, we have an A-linear isomorphism

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\oplus_j \Omega^{\mathcal{E}_j}, \Gamma) \simeq \prod_j \Gamma_{\mathcal{E}_j} \text{ for all } \mathcal{E}_j \in \mathcal{C},\ \Gamma \in \mathrm{L}_R(\mathcal{C}, A)$$

In particular

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\Omega^{\mathcal{E}}, \Gamma) \simeq \Gamma_{\mathcal{E}}$$

is the evaluation in $\mathcal{E} \in \mathcal{C}$, is exact: an element $x \in \Gamma_{\mathcal{E}}$ can be thought of as a natural transformation $\Omega^{\mathcal{E}} \to \Gamma$. which implies that $\mathrm{Ext}^1_{\mathrm{L}_R(\mathcal{C},R)}(\Omega^{\mathcal{E}}, -) = 0$.

By Yoneda's lemma, we obtain a functorial map

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\oplus_j \Omega^{\mathcal{E}_j}, \oplus_k \Omega^{\mathcal{E}_k}) \to \mathrm{Hom}(\oplus_j \mathcal{E}_j, \oplus_k \mathcal{E}_k)$$

which is an isomorphism onto the set of locally finite maps.

Let $\mu \colon \oplus_j \mathcal{E}_j \to \mathcal{E}$ be a surjective map and set $\Delta = \operatorname{Im}(\Omega^\mu)$. Then $\Delta_{\mathcal{E}'}$ is the set of maps $\mathcal{E}' \to \mathcal{E}$ which factors through μ via a locally finite map $\mathcal{E}' \to \oplus_j \mathcal{E}_j$. Given a map $u \colon \overline{\mathcal{E}} \to \mathcal{E}$ in $\mathcal{C}$ we set $u^{-1}(\Delta) = \Delta \times_{\Omega^{\mathcal{E}}} \Omega^{\overline{\mathcal{E}}} \subseteq \Omega^{\overline{\mathcal{E}}}$: $u^{-1}(\Delta)_{\mathcal{E}'}$ is the set of maps $\mathcal{E}' \to \overline{\mathcal{E}}$ such that $\mathcal{E}' \to \overline{\mathcal{E}} \to \mathcal{E}$ factors through μ via a locally finite map $\mathcal{E}' \to \oplus_j \mathcal{E}_j$. Notice that if $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ then $u^{-1}(\Delta) \in \Phi_{\mathcal{C}}(\overline{\mathcal{E}})$. Indeed applying the last remark in 4.4 on the maps $\overline{\mathcal{E}} \to \mathcal{E}$ and $\mu \colon \bigoplus_j \mathcal{E}_j \to \mathcal{E}$ and denoting by $\mathcal{H}$ their fiber product we obtain a surjective map $\bigoplus_s \mathcal{E}_s \to \overline{\mathcal{E}}$ factoring through $\mathcal{H}$ and such that each $\mathcal{E}_s \to \overline{\mathcal{E}}$ belongs to $u^{-1}(\Delta)_{\mathcal{E}_s}$. It follows that the obvious map

$$\mu' \colon \bigoplus_{\tilde{\mathcal{E}} \in \mathcal{C}} \bigoplus_{\omega \in u^{-1}(\Delta)_{\tilde{\mathcal{E}}}} \tilde{\mathcal{E}} \to \mathcal{E}$$

is surjective and clearly $u^{-1}(\Delta) = \operatorname{Im}(\Omega^{\mu'})$.

Lemma 5.3. *Let $\mu \colon \oplus_j \mathcal{E}_i \to \mathcal{E}$ be a surjective map with $\mathcal{E}_j, \mathcal{E} \in \mathcal{C}$ and $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$. Then there is an exact sequence of A-modules*

$$0 \to \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\Omega^{\mathcal{E}}/\operatorname{Im}(\Omega^\mu), \Gamma) \to \Gamma_{\mathcal{E}} \to \prod_j \Gamma_{\mathcal{E}_j}$$

If $\mathcal{T} : \oplus_k \mathcal{E}_k \to \oplus_j \mathcal{E}_j \to \mathcal{E} \to 0$ is a test sequence and Γ is exact on $\mathcal{T}$ then Γ is cohomologically left exact on μ. The converse holds if the map

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\operatorname{Ker}(\Omega^\mu), \Gamma) \to \prod_k \Gamma_{\mathcal{E}_k}$$

obtained applying $\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(-, \Gamma)$ to the map $\oplus_k \Omega^{\mathcal{E}_k} \to \operatorname{Ker}(\Omega^\mu)$ is injective.

Proof. Set $\Delta = \mathrm{Im}(\Omega^\mu)$, $K = \mathrm{Ker}(\Omega^\mu)$. Consider the diagram

$$\begin{array}{ccccccccc}
\mathrm{Hom}(\Omega^{\mathcal{E}}/\Delta,\Gamma) & \hookrightarrow & \Gamma_{\mathcal{E}} & \longrightarrow & \mathrm{Hom}(\Delta,\Gamma) & \longrightarrow & \mathrm{Ext}^1(\Omega^{\mathcal{E}}/\Delta,\Gamma) & \longrightarrow & \mathrm{Ext}^1(\Omega^{\mathcal{E}},\Gamma) \\
 & & & \searrow^{\alpha} & \big\uparrow & & & & \\
 & & & & \prod_j \Gamma_{\mathcal{E}_j} & & & & \\
 & & & & \big\downarrow & \searrow^{\beta} & & & \\
 & & & & \mathrm{Hom}(K,\Gamma) & \xrightarrow{\lambda} & \prod_k \Gamma_{\mathcal{E}_k} & &
\end{array}$$

The convention here is that β and λ are defined only when a test sequence $\mathcal{T}$ as in the statement exists and we will not use them for the first statement. All the other maps are obtained splitting $\Omega^{\mathcal{E}_j} \to \Omega^{\mathcal{E}} \to 0$ into two exact sequences and applying $\mathrm{Hom}(-,\Gamma)$, so that the first line and the central column are exact. The map α obtained as composition is the map defined in the first sequence in the statement. In particular the first claim follows. So let's focus on the second one. The map λ is the second map in the statement while the map β together with α are the maps defining the sequence (4.2). Since $\mathrm{Ext}^1(\Omega^{\mathcal{E}},\Gamma) = 0$ also the second claim follows. ☐

Lemma 5.4. *Let $\Gamma, K \in \mathrm{L}_R(\mathcal{C},R)$ and $u\colon \oplus_q \Omega^{\mathcal{E}_q} \to K$ be a map, where $\mathcal{E}_q \in \mathcal{C}$. If for all $\Omega^{\mathcal{E}} \to K$ with $\mathcal{E} \in \mathcal{C}$ there exists a surjective map $v\colon \oplus_t \mathcal{E}_t \to \mathcal{E}$ with $\mathcal{E}_t \in \mathcal{C}$ such that the composition $\oplus_t \Omega^{\mathcal{E}_t} \to \Omega^{\mathcal{E}} \to K$ factors through u and Γ is cohomologically left exact on v then the map*

$$\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(K,\Gamma) \to \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(\oplus_q \Omega^{\mathcal{E}_q},\Gamma) \simeq \prod_q \Gamma_{\mathcal{E}_q}$$

is injective.

Proof. Let $\mathcal{E} \in \mathcal{C}$ and $x \in K_{\mathcal{E}}$, which corresponds to a map $\Omega^{\mathcal{E}} \to K$. Consider the data given by hypothesis with respect to this last map. We have commutative diagrams

$$\begin{array}{ccc}
\oplus_t \Omega^{\mathcal{E}_t} & \xrightarrow{\Omega^v} & \Omega^{\mathcal{E}} \\
\big\downarrow & & \big\downarrow x \\
\oplus_q \Omega^{\mathcal{E}_q} & \xrightarrow{u} & K
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Hom}(K,\Gamma) & \xrightarrow{\gamma} & \Gamma_{\mathcal{E}} \\
\big\downarrow \lambda & & \big\downarrow \delta \\
\prod_q \Gamma_{\mathcal{E}_q} & \longrightarrow & \prod_t \Gamma_{\mathcal{E}_t}
\end{array}$$

where the second diagram is obtained by applying $\mathrm{Hom}(-,\Gamma)$ to the first one. The map λ is the map in the statement, while γ is the evaluation in $x \in K_{\mathcal{E}}$. Thanks to 5.3 and since Γ is cohomologically left exact on v the map δ is injective. So if $\phi \in \mathrm{Hom}(K,\Gamma)$ is such that $\lambda(\phi) = 0$ it follows that $\gamma(\phi) = \phi_{\mathcal{E}}(x) = 0$, as required. ☐

Theorem 5.5. *If $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ then $\mathrm{Lex}_R(\mathcal{C},A)$ coincides with the subcategory of $\mathrm{L}_R(\mathcal{C},A)$ of cohomologically left exact functors.*

Proof. Let $\mu = \oplus_j \mu_j\colon \oplus_{j\in J} \mathcal{E}_j \to \mathcal{E}$ be a surjective map with $\mathcal{E}, \mathcal{E}_j \in \mathcal{C}$ and set $\Delta = \mathrm{Im}(\Omega^\mu)$, $K = \mathrm{Ker}(\Omega^\mu)$. By 4.4 there exists a test sequence $\oplus_k \mathcal{E}_k \to \oplus_j \mathcal{E}_j \to \mathcal{E} \to 0$. Using 5.3 we have to prove that if $\Gamma \in \mathrm{L}_R(\mathcal{C},R)$ is cohomologically left exact then $\lambda\colon \mathrm{Hom}(K,\Gamma) \to \prod_k \Gamma_{\mathcal{E}_k}$ is injective. We are going to apply 5.4 with respect to the map $\oplus_k \Omega^{\mathcal{E}_k} \to K$. If $\overline{\mathcal{E}} \in \mathcal{C}$, a map $\Omega^{\overline{\mathcal{E}}} \to K$ is a locally finite map $\overline{\mathcal{E}} \to \oplus_j \mathcal{E}_j$ which is zero composed by μ, or, equivalently, mapping in the image of $\oplus_k \mathcal{E}_k \to \oplus_j \mathcal{E}_j$. We apply the last remark in 4.4 to the maps $\oplus_k \mathcal{E}_k \to \oplus_j \mathcal{E}_j$ and $\overline{\mathcal{E}} \to \oplus_j \mathcal{E}_j$. If $\mathcal{H}$ is their fiber product there is a surjective map $\oplus_s \mathcal{E}_s \to \mathcal{H}$ with $\mathcal{E}_t \in \mathcal{C}$ such that $\oplus_s \mathcal{E}_s \to \oplus_k \mathcal{E}_k$ is locally finite and $\oplus_s \mathcal{E}_s \to \overline{\mathcal{E}}$ is surjective. This gives the desired factorization for applying 5.4. ☐

We now show how to reduce the number of test sequences in order to check when a $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ belongs to $\mathrm{Lex}_R(\mathcal{C}, A)$. The following is the key lemma:

Lemma 5.6. *Let $\Phi' \subseteq \Phi_{\mathcal{C}}$ such that, for all $\mathcal{E} \in \mathcal{C}$ and $\Delta \in \Phi_{\mathcal{C}}(\mathcal{E})$ there exists $\Delta' \in \Phi' \cap \Phi_{\mathcal{C}}(\mathcal{E})$ such that $\Delta' \subseteq \Delta$ (inside $\Omega^{\mathcal{E}}$). If $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$, $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ and Γ is cohomologically left exact on all the elements of Φ' then Γ is cohomologically left exact.*

Proof. Consider $\Delta \in \Phi_{\mathcal{C}}(\mathcal{E})$, $\Delta' \subseteq \Delta$ with $\Delta' \in \Phi'$ and the exact sequence $0 \to \Delta/\Delta' \to \Omega^{\mathcal{E}}/\Delta' \to \Omega^{\mathcal{E}}/\Delta \to 0$. Applying $\mathrm{Hom}(-, \Gamma)$, the only non trivial vanishing to check is $\mathrm{Hom}(\Delta/\Delta', \Gamma) = 0$. In other words we need to prove that

$$\mathrm{Hom}(\Delta, \Gamma) \to \mathrm{Hom}(\Delta', \Gamma)$$

is injective. Write $\Delta' = \mathrm{Im}(\Omega^u)$, where $u\colon \oplus_q \mathcal{E}_q \to \mathcal{E}$. Since $\bigoplus_q \Omega^{\mathcal{E}_q} \to \Delta'$ is surjective it is enough to show that $\mathrm{Hom}(-, \Gamma)$ maps $\Omega^u\colon \bigoplus_q \Omega^{\mathcal{E}_q} \to \Delta$ to an injective map. We are going to prove that the hypothesis of 5.4 for $K = \Delta$ are met. If $\Omega^{\mathcal{E}'} \to \Delta \subseteq \Omega^{\mathcal{E}}$ is a map corresponding to $\psi\colon \mathcal{E}' \to \mathcal{E}$, then $\psi^{-1}(\Delta') \in \Phi_{\mathcal{C}}$ and, by hypothesis, we can find $\Phi' \ni \mathrm{Im}(\Omega^v) \subseteq \psi^{-1}(\Delta')$; the last inclusion tells us that v is the factorization required for 5.4. Moreover Γ is cohomologically left exact on v by hypothesis. □

Proposition 5.7. *Let $\mathcal{D} \subseteq \mathrm{QCoh}\, \mathcal{X}$ be a subcategory and $\Gamma \in \mathrm{L}_R(\mathcal{D}, A)$. If $\Gamma \in \mathrm{Lex}_R(\mathcal{D}, A)$ then Γ is exact on finite test sequences in $\mathcal{D}$ and transforms any arbitrary direct sum of objects of $\mathcal{D}$ and which belongs to $\mathcal{D}$ into a product. The converse holds if one of the following conditions is satisfied:*

- *the category $\mathcal{D}$ is stable by arbitrary direct sums;*
- *all the sheaves in $\mathcal{D}$ are finitely presented, $\mathcal{D} \subseteq \mathrm{QCoh}^{\mathcal{D}} \mathcal{X}$ and $\mathcal{X}$ is quasi-compact. In this case $\Gamma \in \mathrm{Lex}_R(\mathcal{D}, A)$ if and only if it is cohomologically left exact on all surjective maps $\mathcal{E}' \to \mathcal{E}$ with $\mathcal{E} \in \mathcal{D}$ and $\mathcal{E}' \in \mathcal{D}^{\oplus}$.*

In any of the above cases, if moreover $\mathcal{D}$ is additive and all surjections in $\mathcal{D}$ have kernel in $\mathcal{D}$ then $\mathrm{Lex}_R(\mathcal{D}, A)$ is the subcategory of $\mathrm{L}_R(\mathcal{D}, A)$ of functors which are left exact on short exact sequences in $\mathcal{D}$ and transforms arbitrary direct sums in products.

Proof. If $\Gamma \in \mathrm{Lex}_R(\mathcal{D}, A)$ then it is clearly exact on finite test sequences. Given a set $\{\mathcal{E}_j \in \mathcal{D}\}_{j \in J}$ set $\mathcal{E} = \bigoplus_j \mathcal{E}_j$. If $\mathcal{E} \in \mathcal{D}$, then the sequence

$$0 \to \bigoplus_{j \in J} \mathcal{E}_j \to \mathcal{E} \to 0$$

is a test sequence and therefore we get that the natural map $\Gamma_{\mathcal{E}} \to \prod_j \Gamma_{\mathcal{E}_j}$ is an isomorphism. Moreover the last part of the statement follows easily from the first part. Finally if $\mathcal{D}$ is stable by arbitrary direct sums is easy to see that the converse holds.

So we focus on the second point and we assume that all the sheaves in $\mathcal{D}$ are finitely presented, $\mathcal{D} \subseteq \mathrm{QCoh}^{\mathcal{D}} \mathcal{X}$ and that $\mathcal{X}$ is quasi-compact. Since the class of finitely presented quasi-coherent sheaves on $\mathcal{X}$ modulo isomorphism is a set, we can assume $\mathcal{D} = \mathcal{C}$ small. Let $\Phi' \subseteq \Phi_{\mathcal{C}}$ be the subset of functors of the form $\mathrm{Im}(\Omega^{\mu})$ for some surjective map $\mu\colon \mathcal{E}' \to \mathcal{E}$ with $\mathcal{E} \in \mathcal{C}$ and $\mathcal{E}' \in \mathcal{C}^{\oplus}$. The set Φ' satisfies the hypothesis of 5.6: if $v\colon \oplus_{j \in J} \mathcal{E}_j \to \mathcal{E}$ is a surjective map then there exists a finite subset $J_0 \subseteq J$ such that $v_{|\mathcal{E}'}\colon \mathcal{E}' = \oplus_{j \in J_0} \mathcal{E}_j \to \mathcal{E}$ is surjective because $\mathcal{E}$ is of finite type and $\mathcal{X}$ is quasi-compact. In particular, taking into account 5.5, the last claim of the second point follows. It remains to show that if $\Gamma \in \mathrm{L}_R(\mathcal{C}, A)$ is exact on finite test sequences then Γ is cohomologically left exact on all the elements of Φ'. Let $\mu\colon \mathcal{E}' \to \mathcal{E}$ be a surjective map with $\mathcal{E} \in \mathcal{C}$ and $\mathcal{E}' \in \mathcal{C}^{\oplus}$. Since $\mathcal{E}$ is finitely presented and $\mathcal{E}'$ is of finite type it follows that $\mathrm{Ker}(\mu)$ is of finite type and, since $\mathcal{X}$ is quasi-compact and $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$, there exists $\mathcal{E}'' \in \mathcal{C}^{\oplus}$ and a surjective map $\mathcal{E}'' \to \mathrm{Ker}(\mu)$. Thus $\mathcal{E}'' \to \mathcal{E}' \to \mathcal{E} \to 0$ is a finite test sequence and by 5.3 it follows that Γ is cohomologically left exact on μ as required. □

There is another characterization of $\mathrm{Lex}_R(\mathcal{C}, A)$ in terms of sheaves on a site. Although we will not use it in this paper, I think it is worth to point out. We refer to [Aut19, Tag 00YW] for general definitions and properties. We start by comparing $\mathrm{Lex}_R(\mathcal{C}, A)$ and $\mathrm{Lex}_R(\mathcal{C}^\oplus, A)$.

Proposition 5.8. *If $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ then the equivalence $\mathrm{L}_R(\mathcal{C}^\oplus, A) \simeq \mathrm{L}_R(\mathcal{C}, A)$ maps $\mathrm{Lex}_R(\mathcal{C}^\oplus, A)$ to $\mathrm{Lex}_R(\mathcal{C}, A)$.*

Proof. We can assume $A = R$. Let $\Gamma \in \mathrm{L}_R(\mathcal{C}^\oplus, R)$ such that $\Gamma \in \mathrm{Lex}_R(\mathcal{C}, R)$ and consider $\Phi' \subseteq \Phi_{\mathcal{C}^\oplus}$ the set of subfunctors $\Delta \subseteq \Omega^{\mathcal{E}}$ that can be written as follows: $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r$ and there are surjective maps $\mu_k : \oplus_q \mathcal{E}_{q,k} \to \mathcal{E}_k$ for $\mathcal{E}_k, \mathcal{E}_{q,k} \in \mathcal{C}$ such that $\Delta = \mathrm{Im}(\Omega^{\mu_1}) \oplus \cdots \oplus \mathrm{Im}(\Omega^{\mu_r})$. Since for such Δ we have

$$\Omega^{\mathcal{E}}/\Delta \simeq \oplus_i (\Omega^{\mathcal{E}_i}/\Delta_i)$$

it follows that Γ is cohomologically left exact on all the elements of Φ'. Taking into account 5.5, in order to conclude that $\Gamma \in \mathrm{Lex}_R(\mathcal{C}^\oplus, R)$ we can show that $\Phi' \subseteq \Phi_{\mathcal{C}^\oplus}$ satisfies the hypothesis of 5.6. So let $\Delta = \mathrm{Im}(\Omega^\mu) \in \Phi_{\mathcal{C}^\oplus}$ where $\mu \colon \oplus_q \mathcal{E}_q \to \mathcal{E}$ where $\mathcal{E}, \mathcal{E}_q \in \mathcal{C}^\oplus$. If $\mathcal{E} = \mathcal{E}'_1 \oplus \cdots \oplus \mathcal{E}'_r$ with $\mathcal{E}'_i \in \mathcal{C}$ and $\psi_i \colon \mathcal{E}'_i \to \mathcal{E}$ are the inclusions then $\Delta_i = \psi_i^{-1}(\Delta) \in \Phi_{\mathcal{C}^\oplus}(\mathcal{E}'_i) = \Phi_{\mathcal{C}}(\mathcal{E}'_i)$ and it is easy to see that $\Phi' \ni \Delta_1 \oplus \cdots \oplus \Delta_r \subseteq \Delta$ as required. □

Theorem 5.9. *Assume $\mathcal{C} \subseteq \mathrm{QCoh}^{\mathcal{C}} \mathcal{X}$ and let $\mathcal{J}$ be the smallest Grothendieck topology on $\mathcal{C}^\oplus$ containing $\Phi_{\mathcal{C}^\oplus}$ then $\mathrm{Lex}_R(\mathcal{C}^\oplus, A)$ is the category of sheaves of A-modules on $(\mathcal{C}^\oplus, \mathcal{J})$ which are R-linear. In other words $\Gamma \in \mathrm{L}_R(\mathcal{C}^\oplus, A)$ is left exact if and only if it a sheaf on $(\mathcal{C}^\oplus, \mathcal{J})$.*

Proof. We can assume $A = R$ and $\mathcal{C} = \mathcal{C}^\oplus$. If $\Delta \subseteq \Omega^{\mathcal{E}}$ is a sieve and $f \colon \mathcal{E}' \to \mathcal{E}$ we set $f^{-1}(\Delta) = \Delta \times_{\Omega^{\mathcal{E}}} \Omega^{\mathcal{E}'} \subseteq \Omega^{\mathcal{E}'}$. Let $\widetilde{\mathcal{J}}$ be the set of sieves $\Delta \subseteq \Omega^{\mathcal{E}}$ of $\mathcal{C}$ such that, for all $\Gamma \in \mathrm{Lex}_R(\mathcal{C}, R)$ and maps $f \colon \mathcal{E}' \to \mathcal{E}$ the map

$$\mathrm{Hom}_{(\mathrm{Sets})}(\Omega^{\mathcal{E}'}, \Gamma) \to \mathrm{Hom}_{(\mathrm{Sets})}(f^{-1}(\Delta), \Gamma)$$

is bijective. Here $\mathrm{Hom}_{(\mathrm{Sets})}$ denotes the set of natural transformation of functors with values in (Sets). The set $\widetilde{\mathcal{J}}$ is a Grothendieck topology on $\mathcal{C}$ such that all functors in $\mathrm{Lex}_R(\mathcal{C}, R)$ are sheaves: see [Aut19, Tag 00Z9]. Notice that, by 2.17, if $A, B \in \mathrm{L}_R(\mathcal{C}, R)$ then $\mathrm{Hom}_{(\mathrm{Sets})}(A, B) = \mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(A, B)$. Moreover if $\Delta \in \Phi_{\mathcal{C}}$ and $f \colon \mathcal{E}' \to \mathcal{E}$ is a map in $\mathcal{C}$ then $\Delta' = f^{-1}(\Delta) \in \Phi_{\mathcal{C}}$. If $\Gamma \in \mathrm{L}_R(\mathcal{C}, R)$ then, applying $\mathrm{Hom}_{\mathrm{L}_R(\mathcal{C},R)}(-, \Gamma)$ on the exact sequence $0 \to \Delta' \to \Omega^{\mathcal{E}'} \to \Omega^{\mathcal{E}'}/\Delta' \to 0$ and taking into account that $\mathrm{Ext}^1(\Omega^{\mathcal{E}'}, \Gamma) = 0$ we obtain an exact sequence

$$0 \to \mathrm{Hom}(\Omega^{\mathcal{E}'}/\Delta', \Gamma) \to \mathrm{Hom}_{(\mathrm{Sets})}(\Omega^{\mathcal{E}'}, \Gamma) \to \mathrm{Hom}_{(\mathrm{Sets})}(\Delta', \Gamma) \to \mathrm{Ext}^1(\Omega^{\mathcal{E}'}/\Delta', \Gamma) \to 0$$

By 5.5 $\Phi_{\mathcal{C}} \subseteq \widetilde{\mathcal{J}}$ and therefore $\mathcal{J} \subseteq \widetilde{\mathcal{J}}$, which means that if $\Gamma \in \mathrm{Lex}_R(\mathcal{C}, A)$ then Γ is a sheaf on $(\mathcal{C}, \mathcal{J})$. Conversely if Γ is a sheaf on $(\mathcal{C}, \mathcal{J})$ we immediately see from the above sequence that Γ is cohomologically left exact, which ends the proof. □

6. Some special cases

We now apply 4.12 and 5.7 in some (more) concrete situations. The symbol A will denote an R-algebra.

This is Gabriel-Popescu's theorem for the category $\mathrm{QCoh}\, \mathcal{X}$.

Theorem 6.1. *[Gabriel-Popescu's theorem] Let $\mathcal{X}$ be a pseudo-algebraic fibered category. If $\mathcal{E}$ is a generator of $\mathrm{QCoh}\, \mathcal{X}$ then the functor*

$$\mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, -) \colon \mathrm{QCoh}_A \mathcal{X} \to \mathrm{Mod}_{right}(\mathrm{End}_{\mathcal{X}}(\mathcal{E}) \otimes_R A)$$

is fully faithful and has an exact left adjoint.

Proof. It follows from 4.12 and 4.6 with $\mathcal{D} = \mathcal{C} = \{\mathcal{E}\}$: in this case

$$\mathrm{L}_R(\mathcal{C}, A) \simeq \mathrm{Mod}_{\mathrm{right}}(\mathrm{End}_{\mathcal{X}}(\mathcal{E}) \otimes_R A), \ \Gamma \longmapsto \Gamma_{\mathcal{E}}$$

□

Theorem 6.2. *Let $\mathcal{X}$ be a pseudo-algebraic fibered category over R. The category $\mathrm{Lex}_R(\mathrm{QCoh}\,\mathcal{X}, A)$ is the category of contravariant, R-linear and left exact functors $\Gamma\colon \mathrm{QCoh}\,\mathcal{X} \to \mathrm{Mod}\,A$ which transform arbitrary direct sums into products. Moreover the functor*

$$\Omega^*\colon \mathrm{QCoh}_A\,\mathcal{X} \to \mathrm{Lex}_R(\mathrm{QCoh}\,\mathcal{X}, A)$$

is an equivalence of categories.

Proof. Follows from 4.12 and 5.7 with $\mathcal{D} = \mathrm{QCoh}\,\mathcal{X}$. □

In what follows $\mathrm{QCoh}_{\mathrm{fp}}\,\mathcal{X}$ denotes the category of quasi-coherent sheaves of finite presentation.

Theorem 6.3. *Let $\mathcal{X}$ be a quasi-compact fibered category over R such that $\mathrm{QCoh}_{fp}\,\mathcal{X}$ generates $\mathrm{QCoh}\,\mathcal{X}$ (e.g. a quasi-compact and quasi-separated scheme by* [Gro60, Section 6.9]*). Then $\mathrm{Lex}_R(\mathrm{QCoh}_{fp}\,\mathcal{X}, A)$ is the category of contravariant, R-linear functors $\mathrm{QCoh}_{fp}\,\mathcal{X} \to \mathrm{Mod}\,A$ which are left exact on right exact sequences. Moreover the functors*

$$\Omega^*\colon \mathrm{QCoh}_A\,\mathcal{X} \to \mathrm{Lex}_R(\mathrm{QCoh}_{fp}\,\mathcal{X}, A), \ \mathcal{F}_{*,\mathrm{QCoh}_{fp}(\mathcal{X})}\colon \mathrm{Lex}_R(\mathrm{QCoh}_{fp}\,\mathcal{X}, A) \to \mathrm{QCoh}_A\,\mathcal{X}$$

are quasi-inverses of each other.

Proof. Follows from 4.12 and 5.7 □

When $\mathcal{X}$ is a noetherian algebraic stack then $\mathrm{QCoh}_{\mathrm{fp}}\,\mathcal{X} = \mathrm{Coh}\,\mathcal{X}$ is abelian and generates $\mathrm{QCoh}\,\mathcal{X}$ (see [LMB05, Prop 15.4]). In particular we obtain:

Theorem 6.4. *Let $\mathcal{X}$ be a noetherian algebraic stack. The category $\mathrm{Lex}_R(\mathrm{Coh}\,\mathcal{X}, A)$ is the category of contravariant, R-linear and left exact functors $\mathrm{Coh}\,\mathcal{X} \to \mathrm{Mod}\,A$. Moreover the functor*

$$\Omega^*\colon \mathrm{QCoh}_A\,\mathcal{X} \to \mathrm{Lex}_R(\mathrm{Coh}\,\mathcal{X}, A), \ \mathcal{F}_{*,\mathrm{Coh}(\mathcal{X})}\colon \mathrm{Lex}_R(\mathrm{Coh}\,\mathcal{X}, A) \to \mathrm{QCoh}_A\,\mathcal{X}$$

are quasi-inverses of each other.

Theorem 6.5. *Let $\mathcal{X}$ be a quasi-compact fibered category over R such that $\mathrm{Vect}\,\mathcal{X}$ generates $\mathrm{QCoh}\,\mathcal{X}$. Then $\mathrm{Lex}_R(\mathrm{Vect}\,\mathcal{X}, A)$ is the category of contravariant, R-linear and left exact functors $\mathrm{Vect}\,\mathcal{X} \to \mathrm{Mod}\,A$. Moreover the functors*

$$\Omega^*\colon \mathrm{QCoh}_A\,\mathcal{X} \to \mathrm{Lex}_R(\mathrm{Vect}\,\mathcal{X}, A), \ \mathcal{F}_{*,\mathrm{Vect}(\mathcal{X})}\colon \mathrm{Lex}_R(\mathrm{Vect}\,\mathcal{X}, A) \to \mathrm{QCoh}_A\,\mathcal{X}$$

are quasi-inverses of each other.

Proof. Follows from 4.12 and 5.7, taking into account that all surjections in $\mathrm{Vect}\,\mathcal{X}$ have kernels in $\mathrm{Vect}\,\mathcal{X}$. □

Theorem 6.6. *Let B be an R-algebra and $\mathcal{D} \subseteq \mathrm{Mod}\,B$ be a subcategory that generates $\mathrm{Mod}\,B$, that is there exists $\mathcal{E}_1, \dots, \mathcal{E}_r \in \mathcal{D}$ with a surjective map $\bigoplus_i \mathcal{E}_i \to B$. Then the functor*

$$\Omega^*\colon \mathrm{Mod}(A \otimes_R B) \to \mathrm{Lex}_R(\mathcal{D}, A)$$

is an equivalence of categories. Moreover if $\mathcal{D} \subseteq \mathrm{Vect}\,B$ then $\mathrm{Lex}_R(\mathcal{D}, A) = \mathrm{L}_R(\mathcal{D}, A)$.

Proof. If $\mathcal{X} = \mathrm{Spec}\,B$, then $\mathrm{QCoh}_A\,\mathcal{X} \simeq \mathrm{Mod}(A \otimes_R B)$ and the first part follows from 4.12. For the last claim, observe that any $\Gamma\colon \mathrm{Vect}\,B \to \mathrm{Mod}\,A$ is exact because any short exact sequence in $\mathrm{Vect}\,B$ splits. By 6.5 we can conclude that $\mathrm{L}_R(\mathrm{Vect}\,B, A) = \mathrm{Lex}_R(\mathrm{Vect}\,B, A)$. If now $\mathcal{D} \subseteq \mathrm{Vect}\,B$ and $\Gamma \in \mathrm{L}_R(\mathcal{D}, A)$, we can extend it to $\overline{\Gamma} \in \mathrm{L}_R(\mathrm{Vect}\,B, A)$ and therefore $\Gamma = \overline{\Gamma}_{|\mathcal{D}} \in \mathrm{Lex}_R(\mathcal{D}, A)$. □

We want to extend Theorem 4.12 to functors with monoidal structures.

Definition 6.7. If $\mathcal{D}$ is a monoidal subcategory of $\mathrm{QCoh}\,\mathcal{X}$ we define $\mathrm{PMLex}_R(\mathcal{D},A)$ (resp. $\mathrm{MLex}_R(\mathcal{D},A)$) as the subcategory of $\mathrm{PML}_R(\mathcal{D},A)$ (resp. $\mathrm{ML}_R(\mathcal{D},A)$) of functors Γ such that $\Gamma\in\mathrm{Lex}_R(\mathcal{D},A)$.

Theorem 6.8. *Let $\mathcal{D}$ be a monoidal subcategory of $\mathrm{QCoh}\,\mathcal{X}$ that generates it. Then the functors*

$$\Omega^*\colon \mathrm{Rings}_A\,\mathcal{X}\to\mathrm{PMLex}_R(\mathcal{D},A)\ \text{and}\ \Omega^*\colon \mathrm{QAlg}_A\,\mathcal{X}\to\mathrm{MLex}_R(\mathcal{D},A)$$

(see 2.25) are equivalence of categories. If $\mathcal{D}$ is small a quasi inverse is given by $\mathscr{A}_{*,\mathcal{D}}\colon\mathrm{PMLex}_R(\mathcal{D},A)\to\mathrm{Rings}_A\,\mathcal{X}$ and $\mathscr{A}_{*,\mathcal{D}}\colon\mathrm{MLex}_R(\mathcal{D},A)\to\mathrm{QAlg}_A\,\mathcal{X}$ respectively (see 2.26). Moreover if $\overline{\mathcal{D}}\subseteq\mathcal{D}$ is a monoidal subcategory that generates $\mathrm{QCoh}\,\mathcal{X}$ the restriction functors $\mathrm{PMLex}_R(\mathcal{D},A)\to\mathrm{PMLex}_R(\overline{\mathcal{D}},A)$ and $\mathrm{MLex}_R(\mathcal{D},A)\to\mathrm{MLex}_R(\overline{\mathcal{D}},A)$ are equivalences.

Proof. Assume that $\mathcal{D}$ is small. Then $\Omega^*\colon\mathrm{Rings}_A\,\mathcal{X}\to\mathrm{PMLex}_R(\mathcal{D},A)$ and $\mathscr{A}_{*,\mathcal{D}}\colon\mathrm{PMLex}_R(\mathcal{D},A)\to\mathrm{Rings}_A\,\mathcal{X}$ are quasi-inverses of each other because, by 2.26, we have natural transformations $\mathrm{id}\to\Omega^*\circ\mathscr{A}_{*,\mathcal{D}}$ and $\mathscr{A}_{*,\mathcal{D}}\circ\Omega^*\to\mathrm{id}$ which are isomorphisms thanks to 4.12. In particular if $\overline{\mathcal{D}}\subseteq\mathcal{D}$ is a monoidal subcategory that generates $\mathrm{QCoh}\,\mathcal{X}$ then the restriction functor $\mathrm{PMLex}_R(\mathcal{D},A)\to\mathrm{PMLex}_R(\overline{\mathcal{D}},A)$ is an equivalence. Since γ and β preserve unities by 2.26, the same holds if we replace $\mathrm{Rings}_A\,\mathcal{X}$ by $\mathrm{QAlg}_A\,\mathcal{X}$ and $\mathrm{PMLex}_R(\mathcal{D},A)$ by $\mathrm{MLex}_R(\mathcal{D},A)$.

Now assume that $\mathcal{D}$ is general. Notice that there exists a small subcategory $\mathcal{C}'\subseteq\mathcal{D}$ that generates $\mathrm{QCoh}\,\mathcal{X}$ thanks to 4.11. If $I\subseteq\mathcal{D}$ is a set we set $\mathcal{C}_I\subseteq\mathcal{D}$ for the category whose objects are isomorphic to a (multiple) tensor product with factors in $C\cup I$ and $\mathcal{O}_\mathcal{X}$. We have that $\mathcal{C}_I$ is a collection of small monoidal subcategories of $\mathcal{D}$, that generate $\mathrm{QCoh}\,\mathcal{X}$ and such that $\mathcal{C}_I\subseteq\mathcal{C}_{I'}$ if $I\subseteq I'$. We can show that the restrictions $\mathrm{PMLex}_R(\mathcal{D},A)\to\mathrm{PMLex}_R(\mathcal{C}_\emptyset,A)$ and $\mathrm{MLex}_R(\mathcal{D},A)\to\mathrm{MLex}_R(\mathcal{C}_\emptyset,A)$ are equivalences by proceeding as in the proof of 4.12. All the other claims in the statement follow easily from this fact. ☐

Theorem 6.9. *The theorems 6.2, 6.3, 6.4, 6.5 continue to hold if we replace Lex_R by PMLex_R (resp. MLex_R), $\mathrm{QCoh}_A\,\mathcal{X}$ by $\mathrm{QRings}_A\,\mathcal{X}$ (resp. $\mathrm{QAlg}_A\,\mathcal{X}$), $\mathcal{F}_{*,\mathcal{C}}$ by $\mathscr{A}_{*,\mathcal{C}}$ and the word "functors" by "pseudo-monoidal functors" (resp. "monoidal functors").*

7. Quasi projective schemes and more

The goal of this section is to show the relation between the sheafification functors we defined and the classical one for projective schemes. The dictionary is explained in the following result. In what follows $\mathcal{X}$ is a pseudo-algebraic fibered category over a ring R and A is an R-algebra. If S is a graded algebra we denote by $\mathrm{GMod}(S)$ the category of graded S-modules.

Proposition 7.1. *Let $\mathcal{L}$ be an invertible sheaf over $\mathcal{X}$ and set $\mathcal{C}_\mathcal{L}=\{\mathcal{L}^{\otimes n}\}_{n\in T}$, where T is either $\mathbb{N}$ or $\mathbb{Z}$, and*

$$S_\mathcal{L}=\bigoplus_{n\in T}\mathrm{H}^0(\mathcal{L}^{-\otimes n})$$

with its canonical $\mathrm{H}^0(\mathcal{O}_\mathcal{X})$-algebra structure. Then

$$\mathrm{L}_R(\mathcal{C}_\mathcal{L},A)\to\mathrm{GMod}(S_\mathcal{L}\otimes_R A),\ \Gamma\longmapsto\bigoplus_{n\in T}\Gamma_{\mathcal{L}^{\otimes n}}$$

is well defined and an equivalence of categories. Under this equivalence $\Omega^*\colon\mathrm{QCoh}_A\,\mathcal{X}\to\mathrm{L}_R(\mathcal{C}_\mathcal{L},A)$ become

$$\mathrm{QCoh}_A\,\mathcal{X}\to\mathrm{GMod}(S_\mathcal{L}\otimes_R A),\ \mathcal{G}\longmapsto\bigoplus_{n\in T}\mathrm{H}^0(\mathcal{G}\otimes\mathcal{L}^{-\otimes n})$$

while $\mathcal{F}_{\Gamma,\mathcal{C}}\colon \mathrm{L}_R(\mathcal{C},A) \to \mathrm{QCoh}_A \mathcal{X}$ *become the functor* $\widetilde{-}\colon \mathrm{GMod}(S_\mathcal{L} \otimes_R A) \to \mathrm{QCoh}_A \mathcal{X}$ *defined by*

$$\widetilde{M}(\xi\colon \operatorname{Spec} B \to \mathcal{X}) = (M \otimes_{S_\mathcal{L}} S_{\xi^*\mathcal{L}})_0$$

Proof. Given a T-graded A-module $\bigoplus_n T_n$ a graded $S_\mathcal{L} \otimes_R A$-module structure is given by R-linear maps

$$\mathrm{H}^0(\mathcal{L}^{-\otimes q}) \to \mathrm{Hom}_A(T_m, T_{m+q}) \text{ for } m, q \in T$$

satisfying the obvious conditions. Instead a structure of contravariant R-linear functor $\Gamma\colon \mathcal{C}_\mathcal{L} \to \mathrm{Mod}\, A$ with $\Gamma_{\mathcal{L}^{\otimes n}} = T_n$ is given by maps

$$\mathrm{Hom}_\mathcal{X}(\mathcal{L}^{\otimes(m+q)}, \mathcal{L}^{\otimes m}) \to \mathrm{Hom}_A(T_m, T_{m+q}) \text{ for } m, q \in T$$

again satisfying certain compatibility conditions. Thus one has to check that the isomorphisms

$$\mathrm{H}^0(\mathcal{L}^{-\otimes q}) \to \mathrm{Hom}_\mathcal{X}(\mathcal{L}^{\otimes(m+q)}, \mathcal{L}^{\otimes m})$$

are compatible with tensor products, which is an elementary computation. The description of Ω^* follows from definition because

$$\Omega^\mathcal{G}_{\mathcal{L}^{\otimes n}} = \mathrm{Hom}_\mathcal{X}(\mathcal{L}^{\otimes n}, \mathcal{G}) = \mathrm{H}^0(\mathcal{G} \otimes \mathcal{L}^{-\otimes n})$$

Let $\xi\colon \operatorname{Spec} B \to \mathcal{X}$ be a map. Notice that $\xi^*\mathcal{C}_\mathcal{L} \simeq \mathcal{C}_{\xi^*\mathcal{L}}$ and that there is a canonical graded algebra morphism $S_\mathcal{L} \to S_{\xi^*\mathcal{L}}$. It is clear that the restriction $\xi_*\colon \mathrm{L}_R(\mathcal{C}_{\xi^*\mathcal{L}}, A) \to \mathrm{L}_R(\mathcal{C}_\mathcal{L}, A)$ corresponds to the restriction $\mathrm{GMod}(S_{\xi^*\mathcal{L}} \otimes_R A) \to \mathrm{GMod}(S_\mathcal{L} \otimes_R A)$. It therefore follows that $\xi^*\colon \mathrm{L}_R(\mathcal{C}_\mathcal{L}, A) \to L_R(\mathcal{C}_{\xi^*\mathcal{L}}, A)$ correspond to the tensor product $-\otimes_{S_\mathcal{L}} S_{\xi^*\mathcal{L}}\colon \mathrm{GMod}(S_\mathcal{L} \otimes_R A) \to \mathrm{GMod}(S_{\xi^*\mathcal{L}} \otimes_R A)$. On the other hand one can check easily that

$$\Omega^*\colon \mathrm{QCoh}_A \operatorname{Spec} B = \mathrm{Mod}(B \otimes_R A) \to \mathrm{GMod}(S_{\xi^*\mathcal{L}} \otimes_R A) \simeq \mathrm{L}_R(\mathcal{C}_{\xi^*\mathcal{L}}, A),\ N \longmapsto N \otimes_{(B\otimes_R A)} S_{\xi^*\mathcal{L}}$$

and that this is natural in $\xi \in \mathcal{X}$. Its adjoint is

$$\mathrm{GMod}(S_{\xi^*\mathcal{L}} \otimes_R A) \to \mathrm{Mod}(B \otimes_R A),\ M \longmapsto M_0$$

By 2.11 we obtain the description of $\mathcal{F}_{\Gamma,\mathcal{C}}$. □

Theorem 7.2. *Let* $\mathcal{X}$ *be a pseudo-algebraic fibered category over* R*,* $\mathcal{L}$ *a line bundle on* $\mathcal{X}$ *and assume that* $\mathcal{C}_\mathcal{L} = \{\mathcal{L}^{\otimes n}\}_{n\in T}$ *generates* $\mathrm{QCoh}(\mathcal{X})$*, where* T *is either* $\mathbb{N}$ *or* $\mathbb{Z}$*. Set* $S_\mathcal{L} = \bigoplus_{n\in T} \mathrm{H}^0(\mathcal{L}^{-\otimes n})$*. Then the functors*

$$\Gamma_*\colon \mathrm{QCoh}_A \mathcal{X} \to \mathrm{GMod}(S_\mathcal{L} \otimes_R A),\ \mathcal{G} \longmapsto \Gamma_*(\mathcal{G}) = \bigoplus_{n\in T} \mathrm{H}^0(\mathcal{G} \otimes \mathcal{L}^{-\otimes n})$$

$$\widetilde{-}\colon \mathrm{GMod}(S_\mathcal{L} \otimes_R A) \to \mathrm{QCoh}_A \mathcal{X},\ \widetilde{M}(\xi\colon \operatorname{Spec} B \to \mathcal{X}) = (M \otimes_{S_\mathcal{L}} S_{\xi^*\mathcal{L}})_0$$

are well defined and the first one is right adjoint to the second one. Moreover Ω^* *is fully faithful and left exact,* $\widetilde{-}$ *is exact and the morphism*

$$\widetilde{\Gamma_*(\mathcal{G})} \to \mathcal{G} \text{ for } \mathcal{G} \in \mathrm{QCoh}_A \mathcal{X}$$

is an isomorphism.

Proof. All claims are consequence of 3.8 , 4.6 and 7.1. □

Remark 7.3. If $p\colon \mathcal{Y} \to \mathcal{X}$ is a representable and quasi-affine map of pseudo-algebraic fibered categories and $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ then $p^*p_*\mathcal{F} \to \mathcal{F}$ is surjective. In particular if $\mathcal{C} \subseteq \mathrm{QCoh}(\mathcal{X})$ generates $\mathrm{QCoh}(\mathcal{X})$ then $p^*\mathcal{C} \subseteq \mathrm{QCoh}(\mathcal{Y})$ generates $\mathrm{QCoh}(\mathcal{Y})$.

The last claim follows from the first because if $\bigoplus_j \mathcal{E}_j \to p_*\mathcal{F}$ is a surjective map with $\mathcal{E}_j \in \mathcal{C}$ then

$$\bigoplus_j p^*\mathcal{E}_j \to p^*p_*\mathcal{F} \to \mathcal{F}$$

is a surjective map with $p^*\mathcal{E}_j \in p^*\mathcal{C}$. For the first claim, since the problem is fpqc local on the target, we can assume $\mathcal{X}$ affine, so that $\mathcal{Y}$ would be a quasi-affine scheme. In this situation we have to prove that all quasi-coherent sheaves on $\mathcal{Y}$ are generated by global sections. If $j\colon \mathcal{Y} \to \operatorname{Spec} C$ is an open immersion and $\mathcal{F} \in \operatorname{QCoh}(\mathcal{Y})$ then one find a surjective map $\mathcal{H} \to j_*\mathcal{F}$ from a free sheaf and $j^*\mathcal{H} \to j^*j_*\mathcal{F} \simeq \mathcal{F}$ will be again a surjective map from a free sheaf.

Remark 7.4. Here we describe some situations in which Theorem 7.2 can be applied.

- If $f\colon X \to \mathbb{P}^n_R$ is a quasi-affine map and $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n_R}(-1)$ then $\mathcal{C}_\mathcal{L} = \{\mathcal{L}^{\otimes n}\}_{n\in\mathbb{N}}$ generates $\operatorname{QCoh}(X)$. This follows applying 7.3 to the quasi-affine map $X \to \mathbb{P}^n_R \to \mathbb{P}^n_\mathbb{Z}$ and the analogous and classical result on $\mathbb{P}^n_\mathbb{Z}$.
- If X is a quasi-compact and quasi-projective scheme over R, $j\colon X \to \mathbb{P}^n_R$ is an immersion and $\mathcal{L} = j^*\mathcal{O}_{\mathbb{P}^n_R}(-1)$ then $\mathcal{C}_\mathcal{L} = \{\mathcal{L}^{\otimes n}\}_{n\in\mathbb{N}}$ generates $\operatorname{QCoh}(X)$. This is a particular case of the previous one. Since X is quasi-compact and $\mathbb{P}^n_R$ is quasi-separated it follows that $X \to \mathbb{P}^n_R$ is a quasi-compact immersion. By [Aut19, Tag 01QV] it follows that $X \to \mathbb{P}^n_R$ is quasi-affine.
- If U is a quasi-affine scheme with an action of $\mathbb{G}_m$, $\mathcal{X} = [U/\mathbb{G}_m]$ and $\mathcal{L}$ is the line bundle corresponding to $\mathcal{X} \to \mathrm{B}\,\mathbb{G}_m$ then $\mathcal{C}_\mathcal{L} = \{\mathcal{L}^{\otimes n}\}_{n\in\mathbb{Z}}$ generates $\operatorname{QCoh}(\mathcal{X})$. Indeed applying 7.3 to the quasi-affine map $\mathcal{X} \to \mathrm{B}\,\mathbb{G}_m$ one reduces to $\mathcal{X} = \mathrm{B}\,\mathbb{G}_m$ as a stack over $\operatorname{Spec}\mathbb{Z}$. Quasi-coherent sheaves on $\mathrm{B}\,\mathbb{G}_m$ (over $\operatorname{Spec}\mathbb{Z}$) are all of the form $\bigoplus_{n\in\mathbb{Z}} G_n \otimes \mathcal{L}^{\otimes n}$ for abelian groups G_n. Thus $\mathcal{C}_\mathcal{L}$ generates $\operatorname{QCoh}(\mathcal{X})$.
- If U is a quasi-affine scheme with an action of μ_d, $\mathcal{X} = [U/\mu_d]$ and $\mathcal{L}$ is the d-torsion line bundle corresponding to $\mathcal{X} \to \mathrm{B}\,\mu_d$ then $\mathcal{C}_\mathcal{L} = \{\mathcal{L}^{\otimes n}\}_{n\in\mathbb{N}}$ generates $\operatorname{QCoh}(\mathcal{X})$. Indeed applying 7.3 to the quasi-affine map $\mathcal{X} \to \mathrm{B}\,\mu_d$ one reduces to $\mathcal{X} = \mathrm{B}\,\mu_d$ as a stack over $\operatorname{Spec}\mathbb{Z}$. Quasi-coherent sheaves on $\mathrm{B}\,\mu_d$ (over $\operatorname{Spec}\mathbb{Z}$) are all of the form $\bigoplus_{n=0}^{d-1} G_n \otimes \mathcal{L}^{\otimes n}$ for abelian groups G_n. Thus $\mathcal{C}_\mathcal{L}$ generates $\operatorname{QCoh}(\mathcal{X})$.

8. Group schemes and representations.

Let G be a flat and affine group scheme over R. In this section we want to interpret the results obtained in the case $\mathcal{X} = \mathrm{B}_R\, G$, the stack of G-torsors for the *fpqc* topology, which is a quasi-compact fpqc stack with affine diagonal.

If A is an R-algebra, by standard theory we have that $\operatorname{QCoh}_A \mathrm{B}_R\, G$ is the category $\operatorname{Mod}^G A$ of G-comodules over A. Recall that the regular representation $R[G]$ of G is by definition $\mathrm{H}^0(G, \mathcal{O}_G)$ with the coaction induced by the right action of G on itself. By definition it comes equipped with a morphism of R-algebras $\varepsilon\colon R[G] \to R$ induced by the unit section of G.

Remark 8.1. If $M \in \operatorname{Mod}^G A$ then the composition

$$(M \otimes_R R[G])^G \to M \otimes_R R[G] \xrightarrow{\mathrm{id}\otimes\varepsilon} M$$

is an isomorphism. This follows from [Jan62, 3.4] applied to $G = H$.

We start with a criterion to find a set of generators for $\operatorname{QCoh} \mathrm{B}_R\, G$.

Proposition 8.2. *If the regular representation $R[G]$ is a filtered direct limit of modules $B_i \in \operatorname{Mod}^G R$ which are finitely presented as R-modules then $\{B_i^\vee\}_{i\in I}$ generates $\operatorname{Mod}^G R$.*

Proof. Set $B = R[G]$ and $\varepsilon_i\colon B_i \to R$ for the composition $B_i \to B \xrightarrow{\varepsilon} R$ and let $M \in \operatorname{Mod}^G R$. Since filtered direct limits commute with tensor products and taking invariants, by 8.1 we have that the limit of the maps $(\varepsilon_i \otimes \mathrm{id}_M)_{|(B_i\otimes M)^G}\colon (B_j \otimes M)^G \to M$ is an isomorphism. This means that for any $m \in M$ there exists $i_m \in I$ and an element $\psi_m \in (B_{i_m} \otimes M)^G$ such that

$(\varepsilon_i \otimes \mathrm{id}_M)(\psi_m) = m$. There is a commutative diagram

$$\begin{array}{ccccc} b \otimes m & \longmapsto & (\phi \mapsto \phi(b)m) & & \\ B_i \otimes M & \longrightarrow & \mathrm{Hom}(B_i^\vee, M) & & \psi \\ & \searrow^{\varepsilon_i \otimes \mathrm{id}} & \swarrow & & \swarrow \\ & M & & \psi(\varepsilon_i) & \end{array}$$

and the horizontal map is G-equivariant. Therefore we obtain a $\delta_m \in \mathrm{Hom}^G({B_i}^\vee, M)$ such that $\delta_m(\varepsilon_i) = m$. This implies that the map

$$\bigoplus_{m \in M} \delta_m \colon \bigoplus_{m \in M} B_{i_m}^\vee \to M$$

is surjective and therefore that M is generated by $\{{B_i}^\vee\}_{i \in I}$. □

Remark 8.3. The class $\mathcal{G}_R$ of flat, affine group schemes G over R such that $R[G]$ is a direct limit of modules in $\mathrm{Vect}(\mathrm{B}_R\, G)$ is stable by arbitrary products, projective limits and base change. Moreover by construction contains all groups which are flat, finite and finitely presented over R, i.e. $R[G] \in \mathrm{Vect}(\mathrm{B}_R\, G)$, and thus all profinite groups. Since, over a Noetherian ring, any G-comodule is the union of the sub G-comodules which are finitely generated R-modules (see [Ser68, Prop 2]), we see that $\mathcal{G}_R$ contains all flat groups defined over a Dedekind domain or a field, such as GL_r, SL_r and all diagonalizable groups. Proposition 8.2 tells us that if $G \in \mathcal{G}_R$ then $\mathrm{B}_R\, G$ has the resolution property, that is $\mathrm{Vect}(\mathrm{B}_R\, G)$ generates $\mathrm{QCoh}(\mathrm{B}_R\, G)$.

Let A be an R-algebra. We denote by $\mathrm{Vect}^G A$ the subcategory of $\mathrm{Mod}^G A$ of G-comodules that are locally free of finite rank (projective of finite type) as A-modules, so that $\mathrm{Vect}(\mathrm{B}_R\, G) \simeq \mathrm{Vect}^G R$. We define $\mathrm{QAdd}^G A$ ($\mathrm{QPMon}^G A$, $\mathrm{QMon}^G A$) as the category of *covariant* R-linear (pseudo-monoidal, monoidal) functors $\mathrm{Vect}^G R \to \mathrm{Mod}\, A$. We set $\mathrm{QRings}^G A$ for the category of $M \in \mathrm{Mod}^G A$ with a G-equivariant map $M \otimes_A M \to M$ and $\mathrm{QAlg}^G A$ for the (not full) subcategory of $\mathrm{QRings}^G A$ of commutative R-algebras.

Definition 8.4. The group G is called *linearly reductive* if the functor $(-)^G \colon \mathrm{Mod}^G R \to \mathrm{Mod}\, R$ is exact.

Remark 8.5. If G is linearly reductive then any short exact sequence in $\mathrm{Vect}^G R$ splits. Indeed if $M \to N$ is surjective then $\mathrm{Hom}_R^G(N, M) \to \mathrm{Hom}_R^G(N, N)$ is surjective, yielding a G-equivariant section $N \to M$.

Theorem 8.6. *If* $\mathrm{B}_R\, G$ *has the resolution property then the functors*

$$\mathrm{Mod}^G A \to \mathrm{QAdd}^G A, \ \mathrm{QRings}^G A \to \mathrm{QPMon}^G A, \ \mathrm{QAlg}^G A \to \mathrm{QMon}^G A$$

which maps M *to the functor* $(- \otimes_R M)^G \colon \mathrm{Vect}^G R \to \mathrm{Mod}\, A$ *are well defined, fully faithful and have essential image the subcategory of functors which are left exact on short exact sequences in* $\mathrm{Vect}^G R$*. In particular they are equivalences if* G *is a linearly reductive group.*

Proof. Set $\mathcal{C} = \mathrm{Vect}^G R$. The functor $(-)^\vee \colon \mathrm{Vect}^G R \to \mathrm{Vect}^G R$ is an equivalence and therefore we get equivalences $\mathrm{QAdd}^G A \simeq \mathrm{L}_R(\mathcal{C}, A)$, $\mathrm{QPMon}^G A \simeq \mathrm{PML}_R(\mathcal{C}, A)$ and $\mathrm{QMon}^G A \simeq \mathrm{ML}_R(\mathcal{C}, A)$. Left exact functors are sent to left exact functors. Under those equivalences Ω^M corresponds to $(- \otimes_R M)^G$ because $\mathrm{Hom}_{\mathrm{B}_R\, G}(\mathcal{E}^\vee, M) \simeq \mathrm{H}^0(\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{B}_R\, G}} M) \simeq (\mathcal{E} \otimes_R M)^G$. Thus the result follows from 4.6, 6.5, 6.9 and 8.5. □

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